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THE DISPERSION OF LIGHT IN METALS*

By J. B. Nathanson
[Carnegie Institute of Technology, Pittsburgh, Pa. Received June 25, 1930]

One of the major problems in physics has been that which is concerned with the propagation of radiation in various media. It has been found that the rate of absorption and the velocity of propagation of the radiant energy are dependent upon the nature and physical constitution of the medium, and the frequency of the incident radiation. Stated in a slightly different manner, the refractive index of a given medium has been found to be a function of the radiation frequency. With respect to the degree of absorption of radiation, media have been divided into two general classes, namely, insulators and conductors. The latter absorb light so strongly that they will transmit light only when in the form of very thin films.

An account of the many attempts to explain the dispersion of light would involve a history of modern physics. With each change in our concept of the nature of radiation and matter, came a new explanation of dispersion. One has but to recall that Sellmeier's theory in 1871 was based upon the elastic solid theory of light. It will also be remembered that the presence of electrons in matter was predicted by Helmholtz to explain dispersion, even before their isolation and study by Sir J. J. Thomson. The development of the electron theory by H. A. Lorentz in the early part of this century, has given us the so-called Maxwell-Drude-Lorentz theory of metallic dispersion, while quite recently we find attempts at a quantum explanation of dispersion.

* Retiring presidential address given before the Physical Society of Pittsburgh, May 8, 1930.

THE CLASSICAL THEORY OF METALLIC DISPERSION

The starting point of this theory in the discussion of absorption and propagation of light in matter, is the system of equations known as Maxwell's electromagnetic equations. For an absorbing medium such as a metal, a damped wave is of course found to satisfy these equations. According to the classical picture, the variation of the wave velocity with frequency, i.e., the phenomenon of dispersion, has been attributed to forced vibrations of the electrons within the atoms, under the influence of the harmonically varying electrical force in the incident radiation. These electrons have been presumed to offer both frictional and elastic resistance to their motion. The presence of a frictional term in the equation of motion of the electron, will account for absorption throughout the whole spectral range, while the presence of an elastic force implies the existance of one or more absorption bands.

A solution of the differential equation of motion of the electron leads to a value for its displacement from its equilibrium position, which when substituted back into the equation for the wave motion, yields a value for the wave velocity in the absorbing medium. Comparing this velocity with that which would have been derived had the medium been considered as continuous, and having a dielectric constant K, we find that the latter now emerges as a complex quantity which is a function of the number of electrons per unit volume influencing the propagation of the radiation in the medium, and the frequency of the radiation.

To arrive at a physical interpretation of the meaning of a complex dielectric constant, it must be remembered that according to Maxwell's electromagnetic theory for a continuous medium, the dielectric constant is proportional to the square of the refractive index of the medium. The latter itself must be complex, and hence upon equating real and imaginary parts of the expressions for the dielectric constant and refractive index, we arrive at the following equations expressing the dispersion in an absorbing medium:

$$n^2(1 - k^2) = 1 + \frac{4\pi m(\nu_0^2 - \nu^2)Ne^2}{4\pi^2 m^2(\nu_0^2 - \nu^2)^2 + b^2\nu^2} \tag{1}$$

$$n^2 k = \frac{Ne^2 b\nu}{4\pi^2 m^2(\nu_0^2 - \nu^2)^2 + b^2\nu^2}. \tag{2}$$

In equations (1) and (2), n is the refractive index of the medium, and k is the coefficient of absorption so defined that the intensity of the

radiation in the medium is reduced to ϵ^{-1} of its value after traversing a distance equal to the wave length of the radiation in the medium. N is the number of electrons per unit volume taking part in the dispersion; e and m are the charge and mass of an electron; ν_0 is the natural frequency of a bound electron; ν is the frequency of the radiation, and b is the damping or frictional constant which when multiplied by the velocity of the electron, represents the resistance offered to the motion of the electron. The nature of this damping force has been somewhat obscure. The damping has been ascribed to radiation from the accelerated electron, or perhaps to some collision process involving a loss of momenta of various groups of electrons. The simplest, though doubtful relation of b with other physical quantities, is given by $b = Ne^2/\sigma$, where σ is the electrical conductivity of the medium. Substituting this value of b in (1) and (2), and assuming that both free and bound electrons contribute to the values of the optical constants of the metal, we arrive at equations (3) and (4). The summation sign indicates that there may be several classes of bound electrons, but only one class of free electrons.

$$n^2(1-k^2) = 1 + \Sigma\frac{4\pi m(\nu_0^2-\nu^2)Ne^2}{4\pi^2m^2(\nu_0^2-\nu^2)^2 + N^2e^4\nu^2/\sigma^2} - \frac{4\pi mNe^2}{4\pi^2m^2\nu^2 + N^2e^4/\sigma^2} \quad (3)$$

$$n^2k = \Sigma\frac{N^2e^4\nu/\sigma}{4\pi^2m^2(\nu_0^2-\nu^2)^2 + N^2e^4\nu^2/\sigma^2} + \frac{N^2e^4\nu/\sigma}{4\pi^2m^2\nu^4 + N^2e^4\nu^2/\sigma^2}. \quad (4)$$

From an experimental determination of n and k, and from the character of the dispersion curves, it is possible to locate the absorption bands. Having thus made suitable conclusions about the values of the natural frequencies of the various classes of bound electrons, one might even compute the relative number of free and bound electrons per atom to account for the dispersion curves obtained. The literature of the past twenty five years contains many such attempts.[1] One of the most recent computations has been made by Ruedy.[1] Ne has found that one class of bound electrons is sufficient to account for the dispersion curves of silver and nickel, but that two classes are necessary for

[1] P. Drude, Ann. d. Phys. *14*, 936; 1904. A. Schuster, Phil. Mag. *7*, 151; 1904. J. H. Jeans, Phil. Mag. *17*, 773; 1909. W. Meier, Ann. d. Phys. *31*, 1035; 1910. H. A. Wilson, Phil. Mag. *20*, 835; 1910. J. W. Nicholson, Phil. Mag. *22*, 245 and 266; 1911. R. A. Houstoun, Proc. Roy. Soc. A. *82*, 606; 1909. V. Freedericksz, Ann. d. Phys. *34*, 780; 1911. L. P. Wheeler, Am. J. Sc. *35*, 491; 1913. G. H. Livens, Phil. Mag. *29*, 655; 1915, and 30, 105; 1915. C. V. Kent, Phys. Rev. *22*, 479; 1923. R. Ruedy, Phys. Rev. *32*, 974; 1928.

copper and gold. The number of bound electrons determining the optical properties of these metals seems to be approximately equal to the number of free electrons per cc. The sum of these numbers yields a value which Ruedy assumes as consistant with the total number of valency electrons. From these results and with the help of equations

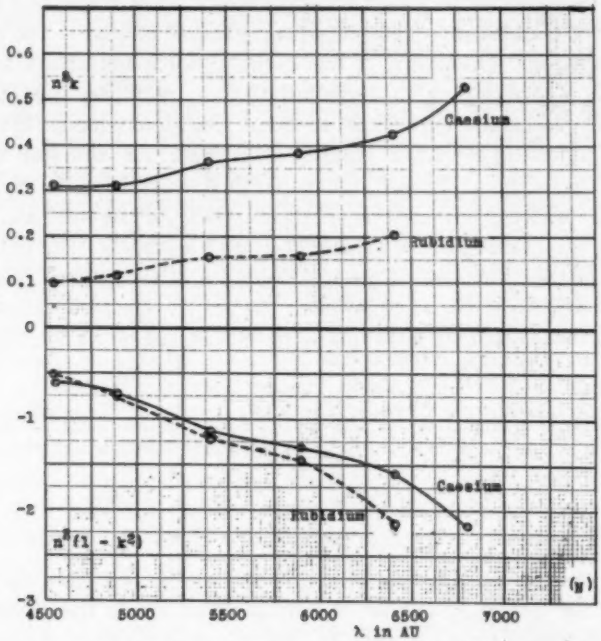

FIG. 1. *Variation with wave length of n^2k and $n^2(1-k^2)$ for rubidium and caesium.*

of the type given by equations (3) and (4), Ruedy then computes the values of $n^2(1-k^2)-1$ and $2n^2k$ as a function of the wave length. In view of the several assumptions made, the agreement between the computed and experimental values of these quantities is quite favorable. Apparently then, the classical theory of dispersion seems to account fairly well for the observed facts.

The results indicate that for regions sufficiently removed from the natural frequencies of any classes of bound electrons, $n^2(1-k^2)$ decreases with the wave length λ, while n^2k increases with λ. This follows from equations (3) and (4), for as ν decreases (or λ increases) the term for the free electrons increases, so that the above quantities vary as

indicated. This type of dispersion in the spectral region 2000 to 7000 AU is also characteristic of such metals as iron, nickel, cobalt, selenium, zinc, bismuth and mercury. One would also expect this for metals having such free electron characteristics as that possessed by the alkalis. This is illustrated by the curves shown in Fig. 1, obtained from a study of rubidium and caesium.[2]

Summarizing, it seems that computations based upon the classical theory of dispersion, require that each atom possess from one to three or four free electrons, depending upon the valence of the metallic element. A more rigorous treatment of the problem of dispersion was attempted by Jeans, Wilson, and others.[1] A statistical analysis of the behavior of various groups of electrons under the influence of the electric vector in the light beam, while more satisfying than the simpler theory of Drude and Lorentz, did not however lead to a more convincing explanation of dispersion in terms of the electron theory.

METALLIC DISPERSION IN THE INFRARED

In the infrared part of the spectrum, Maxwell's electromagnetic theory applied to a continuous medium, is in good agreement with the experimental resuts. The relations between the optical and electrical properties of a metal are given by equations (5) and (6).

$$n^2 = \tfrac{1}{2}(\sqrt{K^2 + 4\sigma^2/\nu^2} + K) \tag{5}$$

$$k^2 = \frac{\sqrt{K^2 + 4\sigma^2/\nu^2} - K}{\sqrt{K^2 + 4\sigma^2/\nu^2} + K}. \tag{6}$$

The reflecting power for normal incidence, assuming a relatively low radiation frequency and a high electrical conductivity of the metal, reduces to equation (7).

$$R = 1 - 2\sqrt{\nu/\sigma} \tag{7}$$

K is the dielectric constant of the metal.

The expression for the reflecting power of a metal in the infrared has been verified experimentally by the well known work of Hagen and Rubens.[2] Of course the relation breaks down in the visible and ultraviolet regions of the spectrum due to our lack of knowledge of the appropriate values of the dielectric constant and electrical con-

[1] J. B. Nathanson, Phys. Rev. *11*, 227; 1918, and *25*, 75; 1925.
[2] E. Hagen and H. Rubens, Ann. d. Phys. *11*, 873; 1903.

ductivities to use at these frequencies. The classical electron theory has attempted to explain the discrepancy with a fair degree of success. As a matter of fact we tried to connect the frictional resistance offered to the movement of the electrons with the ordinary electrical conductivity, using an expression which involved the number of free electrons per cc. We could do no better however, because the mechanism of electrical conduction in solids has always presented a baffling problem. Nevertheless it is clear that dispersion and metallic conduction must be intimately related as shown by the work of Hagen and Rubens. Hence an adequate theory of dispersion in metals must rest upon an adequate theory of conduction.

Within the past few years, Sommerfeld,[4] Bloch, etc., have done much to clarify our ideas of the behavior of electrons in metals. Theoretical expressions have been derived for the electrical properties of metals, which are in better agreement with experiment. The old electron gas theory of conduction has been revivified. Remembering that there are of the order of 10^{22} atoms per cc in a metal, and assuming one or more electrons per atom, it follows that the electron gas is under a tremendous pressure. Under these conditions it has been pointed out that the gas becomes degenerate, and new statistics must be developed to handle the problem of electron motion. This has been accomplished by the statistics of Fermi and Dirac. The distribution-in-energy equations of Maxwell-Botzmann and of Fermi-Dirac yield substantially the same results for a gas at ordinary pressures and temperatures, but differ quite markedly for the electron gas in metals. It might also be added that in the derivation of the expression for the electrical conductivity of a metal, the electrons have lost their purely corpuscular nature, and have assumed the role of wave patterns.

KRONIG'S NEW QUANTUM THEORY OF DISPERSION IN METALS

Recently Kronig[5] has attempted with some success to apply Bloch's[4] ideas of the electrical conductivity of metal crystals, in order to arrive at satisfactory values of the refractive indices and coefficients of absorption of silver and gold. Kronig proceeds to show that equations (5) and (6) will yield the proper values of the optical constants, providing K and σ are properly chosen, i.e., according to Bloch's theory of metallic conduction.

[4] A. Sommerfeld, W. V. Houston, C. Eckart, Z. f. Phys. *47*, 47; 1928. F. Bloch, Z. f. Phys. *52*, 555; 1928.

[5] R. de L. Kronig, Roy. Soc. Proc. *A124*, 409; 1929.

The metal is considered as possessing a cubic lattice. This is the usual concept of the crystal structure of a metal as supported by x-ray evidence. This cubic structure in metals is also illustrated by Fig. 2, which is a micro-photograph* of a rubidium mirror which had turned crystalline after lying undisturbed for several years. The cubical structure of the metal is apparent from the brick like appearance of some of the crystals.

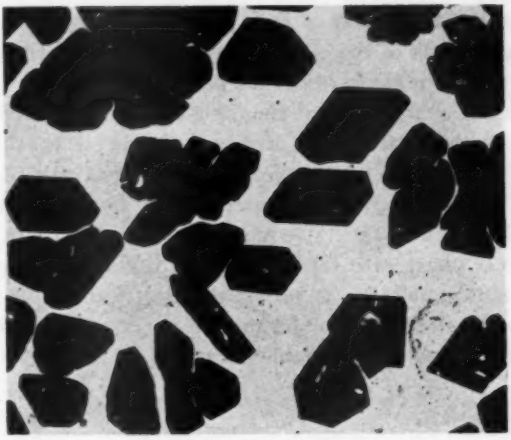

FIG. 2. *Microphotograph of rubidium metal showing crystalline structure. Magnification about 500.*

According to Bloch, the electrons in the metal crystal are considered as being in certain stationary states, and a study is made of the simultaneous influence of the thermal vibrations of the crystal lattice and the varying electromagnetic field upon the distribution of the electrons among their various stationary states. Applying the wave mechanical analysis to these electrons, Bloch found the energy of the various stationary states of the electrons corresponding to the eigen functions. The distribution of the electrons over these states is governed by the statistics of Fermi and Dirac. This distribution will be disturbed when an electric field is applied to these electrons. On account of the thermal vibrations of the lattice, the electrons will experience transitions in state until the proper distribution of stationary states is again attained. This shifting about of the electrons under the influence of the applied field

* The writer is indebted to Dr. V. N. Krivobok for this photograph.

constitutes the electrical current. Bloch shows that the conductivity varies inversely with the absolute temperature as experimentally observed. This is for a constant field.

For a variable electric field, the conductivity and current density will not be related in the same way as for a constant field, since a finite time is required for the new distribution of the electrons to occur over their stationary states. Kronig finds that for good conductors the time of shift or relaxation is between 10^{-13} and 10^{-14} seconds. When the electric field in the incident beam of light has a frequency greater than 10^{13} to 10^{14}, the impacts of the electrons with the lattice, cease to play an important part. From a study of the probability of transitions between the various stationary states at these high frequencies, Kronig arrives at a value for the total current density in the metal, this being proportional to the amplitude of the electric field. That part of the current in phase with the electric field, represents the so called Ohmic current of Maxwell's electromagnetic equations, while that part of the current 90° out of phase with the field, is the displacement current. The former current enables us to obtain the electrical conductivity, while the latter yields the expression for the dielectric constant. These values are given by the rather formidable equations (8) and (9).

$$\sigma = \frac{\pi}{2}\,\Omega\,\frac{1}{\nu}(\nu_2 - \nu)(\nu - \nu_1) \tag{8}$$

$$K = 1 - \frac{e^2\rho^3}{3\pi^3 m a^3 \nu^2} + \frac{\Omega}{\nu^2}\left[\nu_2^2 - \nu_1^2 + \frac{1}{2}(\nu+\nu_1)(\nu+\nu_2)\log\left(\frac{\nu+\nu_1}{\nu+\nu_2}\right)^2 + \frac{1}{2}(\nu-\nu_1)(\nu-\nu_2)\log\left(\frac{\nu-\nu_1}{\nu-\nu_2}\right)^2\right]. \tag{9}$$

As to the meaning of the terms, e and m are the charge and mass of the electron; $1/a^3$ is the number of atoms per cc, which for silver and gold is of the order of 6×10^{22}; ν is the frequency of the incident radiation, where $\nu_1<\nu<\nu_2$. Since the frequency must be large compared to one over the time of relaxation, the computations were confined to the visible and ultraviolet parts of the spectrum so that $\nu_1=5\times10^{14}$ and $\nu_2=2\times10^{15}$.

$\rho^3=3\pi^2Z$ where Z is the number of conducting electrons per atom. The quantity Ω is given by $(e^2P^2/2\pi^3hm^2a^3)(h/4\pi w)^3$, where P is the element of the matrix representing the momentum of an electron and h is Planck's constant. The quantity w is of the order of magnitude of

the mean energy of an electron, and can be evaluated from the expressions $\nu_1 = 4\pi w/h(\pi - \rho)$ and $\nu_2 = 4\pi w/h(\pi + \rho)$.

From these relations Kronig computed the values of K and σ as a function of ν, for the metals silver and gold. He also computed K and σ with the aid of equations (5) and (6), from the experimental values of the optical constants of these elements, as determined by Minor[6] and Meier.[1] The results of all these computations for gold are shown graphically in Fig. 3, where the value of Ω has been taken

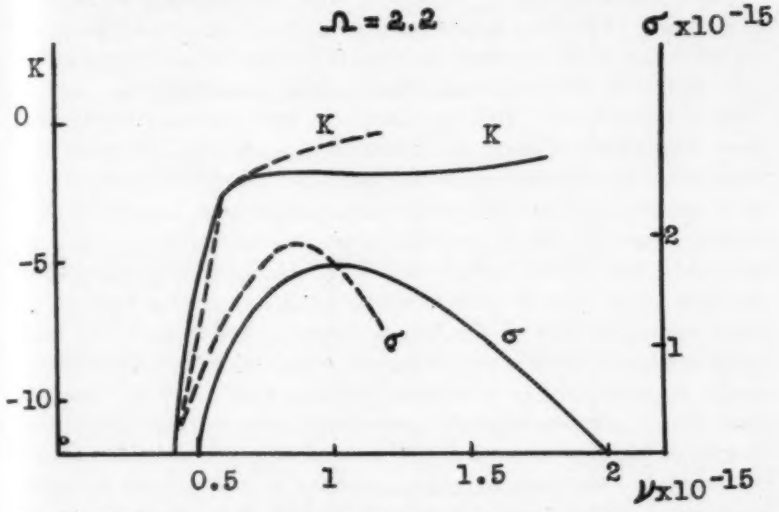

FIG. 3. *Variation with frequency of the dielectric constant and electrical conductivity of gold. Continuous lines represent values according to Kronig's quantum theory. Dotted lines represent values computed from the optical constants.*

as 2.2. The results obtained cannot be expected to check very closely with the experimental values on account of the various simplifying assumptions that have been made. Furthermore optical constants are not physical quantities that can be obtained with a very high degree of precision. However, the curves do show a satisfactory qualitative agreement. It can be said at least, that the new quantum theory of electrons in metals, accounts satisfactorily for the proper magnitudes of the dielectric constant and electrical conductivity in the visible and ultraviolet parts of the spectrum. In contradistinction with the

[6] R. S. Minor, Ann. d. Phys. *10*, 581; 1903.

classical electron theory, only one kind of electron is now considered, no distinction having been made between the free and bound electrons.

THE DISPERSION OF X-RAYS IN METALS

Since x-rays exhibit the same physical characteristics as light, it is therefore desirable to discuss the index of refraction of metals for x-rays. The absorption of x-rays is an extensive subject in itself, hence this discussion will be limited mainly to the refractive index.

The actual refraction of x-rays has been observed only within the past decade. This long looked for phenomenon had escaped previous detection due to the fact that the refractive indices of substances differ only slightly from unity,—and what is more interesting, are only a trifle less than unity. This conclusion was first drawn by Stenstrom[7] from observations of slight deviations from Bragg's law. A refractive index less than unity is however not so unusual, for optical observations have indicated that some metals like silver, magnesium, and the alkalis, possess refractive indices less than unity in the visible part of the spectrum. This at first thought might be interpreted as meaning that the velocity of light in some metals is greater than in a vacuum,—which seems contrary to the special theory of relativity. The true interpretation however is that the phase wave rather than the radiant energy is propagated in a material medium with a velocity greater than in a vacuum, the refractive index being defined as the ratio of the velocity of the phase wave in a vacuum to that in the material medium. Of course in these days of phase velocities as encountered in wave mechanics,—the concept of a velocity greater than that of light in vacuo, is not quite as exciting as in 1898 when Drude[8] made the startling discovery that the refractive index of sodium indicated a velocity of light within it which is 220 times as great as in vacuous space.

It is well known that matter is far more transparent to x-rays than to visible radiations, hence absorption of the type met with in the visible region, can be neglected for x-rays, except in the neighborhood of the critical absorption frequencies of the element. Hence neglecting the frictional term in the classical theory of dispersion, we arrive at an expression for the refractive index which is a modification of equation (1), namely,

$$n = 1 + \Sigma \frac{Ne^2}{2\pi m(\nu_0^2 - \nu^2)}$$

[7] W. Stenstrom, Diss. Lund; 1919.

[8] P. Drude, Ann. d. Phys. *64*, 159; 1898.

$$= 1 - \frac{e^2}{2\pi m}\left[\frac{N_1}{(\nu^2 - \nu_1^2)} + \frac{N_2}{(\nu^2 - \nu_2^2)} + - -\right] \qquad (10)$$

where $N_1 N_2$ etc. are the number of electrons per cc having the natural frequencies ν_1 ν_2 etc. The latter are those given by the x-ray critical absorption limits, and of these only the frequency of the K critical absorption edge is generally of sufficient significance compared with the radiation frequency, to require its consideration in the formula.

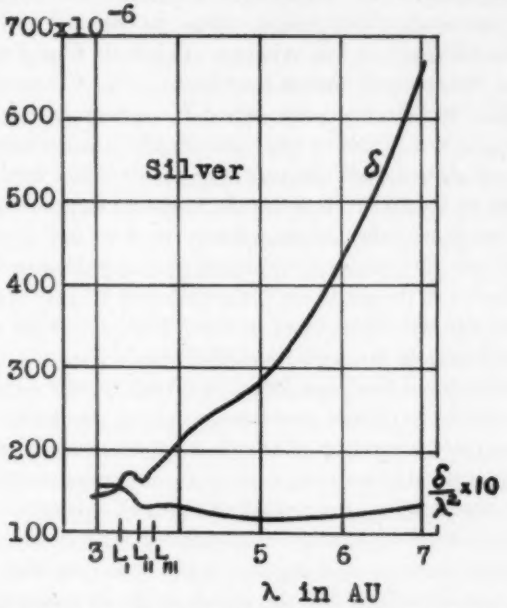

FIG. 4. *Dispersion of soft x-rays in silver. Data according to Dershem.*

Equation 10 is known as the Lorentz formula, and has been experimentally confirmed. For example, Davis and V. Nardroff, and Edwards[9] showed that the agreement between the experimental and theoretical values of n is good providing one assumes a value of 2 for the number of electrons in the K shell of an atom.

An inspection of equation (10) indicates that the refractive index should be somewhat less than unity. This suggests the possibility of total internal reflection at metal-air surfaces, of which there has been ample

[9] H. W. Edwards, Phys. Rev. *30*, 91; 1927. B. Davis and R. von Nardroff, Proc. Nat. Acad. Sc. *10*, 384; 1924.

experimental evidence. Compton[10] and others have used the critical angle method for the determination of refractive indices. The results have been found to agree favorably with the prism method of Davis[11] and others.

Recently the dispersion curves of platinum, silver, calcite and glass have been determined by Dershem,[10] for wave lengths ranging from one to seven AU. He found that $\delta = 1 - n$ increases approximately with the square of the wave length. This holds for regions not too close to critical absorption frequencies. This follows from equation (10), and thus affords an experimental proof of this relation. It follows that δ/λ^2 should be constant for the spectral regions mentioned. Fig. 4 is a graph of Dershem's results for silver, in the region of critical L wave lengths, δ being plotted as a function of the wave length. On the same set of coordinates, are also plotted the values of $(\delta/\lambda^2)\times 10$. The multiplying factor 10 was used in order to bring the values of the ordinates for both graphs to the same order of magnitude for $\lambda = 3$ AU, thus enabling one to compare the relative variations of the ordinates for the two curves. Relative to the graph for δ, the values of δ/λ^2 are approximately constant, even in the region close to the L limits. Similar results were obtained by Dershem for calcite and platinum.

Like results have also been found by Forster[12] who determined the refractive indices for copper sputtered on glass. He found the dispersion curve to hold according to the classical theory right up to either side of the K critical absorption edge, and thus concluded that there is a simple discontinuity in the refractive index at this edge. Previously Hjalmar[13] had detected anomalous dispersion for gypsum and calcite.

The classical theory has given us a fairly satisfying picture of x-ray dispersion, except possibly for the region in the immediate vicinity of a critical absorption edge. It seems as though the electrons in the medium behave as free electrons for the x-ray region—so much so that it is possible with a fair degree of accuracy to compute the refractive index —a computation not possible for the visible part of the spectrum without introducing many qualifying assumptions. The experimental values of the refractive indices check satisfactorily with those computed

[10] A. H. Compton, Bull. Nat. Res. Council, *#20*, 48; 1922. M. de Broglie and J. Thibaud, C. R. *181*, 1034; 1925. E. Dershem, Phys. Rev. *33*, 659; 1929.

[11] B. Davis and C. C. Hatley, Phys. Rev. *23*, 290; 1924. B. Davis and R. von Nardroff, Phys. Rev. *23*, 291; 1924. M. Siegbahn, Jour. de Phys. et le Rad. *6*, 228; 1925. C. M. Slack, Phys. Rev. *27*, 691; 1926.

[12] R. Forster, Helv. Phys. Acta. *1*, 18; 1928.

[13] E. Hjalmar, Ann. d. Phys. *79*, 550; 1926.

by the Lorentz formula, indicating its validity for the x-ray region. Perhaps the good agreement is due to the fact that the natural frequencies need not be known with the precision necessary in the visible and ultaviolet regions, or at least, the critical absorption wave lengths will yield definite information about the values of the natural frequencies.

According to equation(10), we should expect a discontinuity at a natural frequency. We should expect δ to assume large positive values on the short wave-length side of an absorption edge and large negative values on the long wave-length side. Actually all that is seen on Dershem's curves are slight dips. The classical theory does not seem to give the desired variations in the refractive index in the neighborhood of an absorption band. This demands then that the theory must take absorption coefficients into account in order that the character of the dispersion curve may be properly influenced. Richtmyer[14] believes the scattering coefficient is more closely related to the coefficient of absorption of the visible part of the spectrum.

QUANTUM CONCEPTS OF X-RAY DISPERSION

The classical theory of x-ray dispersion is well supported by experiment, hence the theory should be derivable in accordance with quantum principles. The classical picture of an elastically bound electron absorbing energy continously, ought to be brought into harmony with the quantum concepts of stationary states and absorption in quanta. Attempt at a quantum theory of dispersion employing the correspondence principle was made by Wentzel.[15] Kramers[16] and others investigated the behavior of a multiply periodic system subjected to the influence of an harmonically varying external field. He derived an expression for the induced moment of an atom by considering the quantum changes in the stationary states of the atoms. Except for a multiplying factor, this moment was found to be of the same form as that given by the classical theory. An expression for the electrical moment formally identical with Kramer's expression was deduced by Schrodinger[17] on wave mechanical principles. In Kramers' theory the classically bound electrons are replaced by a type of virtual oscillators or

[14] F. K. Richtmyer, Phil. Mag. *4*, 1926; 1927.

[15] G. Wentzel, Z. f. Phys. *29*, 306; 1924.

[16] H. A. Kramers, Nature *113*, 673; 1924, and *114*; 310; 1924. H. A. Kramers and W. Heisenberg, Z. f. Phys. *31*, 681; 1925. J. H. Van Vleck, Phys. Rev. *24*, 344; 1924.

[17] E. Schrodinger, Ann. d. Phys. *81*, 112; 1926.

resonators, whose frequencies are the spectroscopic rather than the orbital ones. One resonator is identified with each possible transition of an electron from a stationary state. It is not to be presumed that the number of Kramers resonators responsible for an absorption band is equal to the number of electrons in the atomic shell corresponding to the critical absorption edge. Prins[18] concludes from a study of the distribution of Kramers resonators based on the wave-length cube rule of absorption, that the data fits the theory best if one assumes a value of 1 to 1.4 K equivalent resonators per atom. Kronig and Kramers[19] concluded similarly that the number of resonators are somewhat less than the number of electrons in the atomic shell.

Kramers quantum theory of dispersion has been extended to the x-ray region by Kronig,[20] and by Kallmann and Mark.[21] The expression for δ derived by Kronig is given by equation (11).

$$\delta = 1 - n = \sum_r \int_0^\infty \frac{Nc\alpha_r(w)dw}{2\pi^2(\nu^2 - w^2)}. \tag{11}$$

The summation is extended over r absorption bands, while the integration with respect to w is carried over each absorption band. $\alpha_r(w)$ is the atomic absorption coefficient whose value is Cc^2Z^4/w^3, where C is a constant; c, the velocity of light in vacuo; Z, the atomic number; ν, the incident frequency; w, the resonance frequency in the absorption band; and N, the number of atoms per unit volume. The integration with respect to w is taken between the limits 0 and ∞, in order to take into account the resonance frequencies within the absorption band as well as terms corresponding to lines associated with the transfer of outer electrons to higher stationary states. The relation for δ holds almost up to the absorption edge.

We have thus a quantum dispersion law for x-rays that takes absorption into account. Kronig[20] quotes some experimental results obtained by Davis and Slack in support of his theory. The observations were made to determine the refractive index of silver on the long wave-length side of the K absorption edge. The results obtained do favor Kronig's quantum theory rather than the Lorentzian theory. On the other hand, Doan[22] determined the refractive index of silver for the Mo $K\alpha_1$ line,

[18] J. A. Prins, Z. f. Phys. *47*, 479; 1928.
[19] R. de L. Kronig and H. A. Kramers, Zeits. f. Phys. *48*, 174; 1928.
[20] R. de L. Kronig, J.O.S.A. & R.S.I. *12*, 551; 1926
[21] H. Kallmann and H. Mark, Ann. d. Phys. *82*, 585; 1927.
[22] R. L. Doan, Phil. Mag. *4*, 100; 1927.

and concluded that his results favor the Lorentz rather than the Kronig formula. Observations to decide between these theories must be made close to the critical absorption edge of an element. Unfortunately the data is rendered somewhat uncertain by the high absorption of the incident radiation. More dispersion data in the vicinity of an absorption edge are therefore desirable.

In conclusion it might be said that if the discussion of dispeesion in rmetals, produces a feeling of uncertainty in one's mind, it is du to the fact that dispersion is but one aspect of the larger problem confronting the physicist today,—namely the problem of the interaction of matter and radiation. That is still the challenging problem of our time.

CONTEMPORARY THEORIES OF LIGHT*

By W. F. G. Swann

[Bartol Research Foundation, Franklin Institute, Swarthmore, Pa. Received June 10, 1930]

Introduction

In the preface of the 1904 edition of his famous theory of "Optics," Sir Arthur Schuster writes as follows:

"There is at present no theory of Optics in the sense that the elastic solid theory was accepted fifty years ago. We have abandoned that theory, and learned that the undulations of light are electromagnetic waves differing only in linear dimensions from the disturbances which are generated by oscillating electric currents or moving magnets. But so long as the character of the displacements which constitute the waves remains undefined we cannot pretend to have established a theory of light. This limitation of our knowledge, which in one sense is a retrogression from the philosophic standpoint of the founders of the undulatory theory, is not always sufficiently recognized and sometimes deliberately ignored. Those who believe in the possiblity of a mechanical conception of the universe and are not willing to abandon the methods which from the time of Galileo and Newton have uniformly and exclusively led to success, must look with the gravest concern on a growing school of scientific thought which rests content with equations correctly representing numerical relationships between different phenomena, even though no precise meaning can be attached to the symbols used."

And now, twenty-five years later, far from having realized what would then have been regarded as a mechanistic theory of light, we have reached a stage of abstract thinking in the matter, in which the electromagnetic equations, whose abstractness Schuster deplored, seem in comparison a very haven of reality to which the most conservative would be willing to return, and there rest in mental peace forever if only the laws of nature would let him.

If we dissect the significance of that desire for reality, for a material explanation, in a modern scientific theory, we usually find that it crystallizes itself into a wish to associate the phenomenon concerned with the actions of something with which we are familiar; and it frequently happens that our familiarity rests upon properties of that something

* An address given before the Optical Society of America, on the occasion of the Thomas Young Memorial Meeting, at Cornell University, Oct. 17, 1929.

which play no part in the field of science we wish to discuss. And so I am moved, at the beginning of this address, to make a slight digression and talk for a moment about what I have called the "Irrevalence of the Obvious." Suppose that I set a student a problem concerning a ball which is hurled vertically upwards in the air with a certain velocity, and I inquire the time at which it will arrive at an assigned altitude. I can well understand the student asking me what the weight of the ball is. I tell him the weight doesn't matter, but he does not like that answer because some of the reality of the ball has vanished from his vision with the weight. He asks me for the color, and I tell him *that* doesn't matter, and I add to his troubles by telling him that I will withdraw even my remark that it was a ball, and leave its shape indefinite. Then, if he is over-materialistically minded he will explode entirely and demand to know how he is to work out *any* problem about the body if I won't tell him its color, shape or mass. There is nothing left for him to think about, and he may well claim that it is difficult for the human mind to think at all unless it has something to think about. Well, to please him, I tell him that the body is red, it weighs ten pounds and is really a round ball. Now he is happy. He takes his paper and pencil, draws the round ball, puts a 10 inside it, paints it red in his mind's eye, and works out the problem. When he brings me the result, I inquire at what point the redness of the ball came into his calculations. He looks through them and finds it didn't come in at all. The result would have been the same for a blue ball. Then I ask him where the 10 pounds came in. He looks again and finds he did not use it; or, if he did, it cancelled out, so that the result would have been the same for a fifty pound ball. Finally, I ask him where the roundness came in, and he finds he did not use that at all. So, I say to him, "Don't ask me for a lot of unnecessary things again." But I think I hear you sympathizing with the poor student. "What harm" you say, "did the redness of the ball do? Why did he sin in thinking it was ten pounds in weight and that it was round, if after all, these things did not matter?" "Well," I agree, that in this particular case the redness did no harm. But I suspect that if I let the student think the ball is red he will come to me some day with some ideas founded purely upon its redness. He will be troubled because he will want some other ball to do the same sort of thing that this ball did, and will be unable to satisfy himself because, perchance, the second ball is blue. Then, I shall have to go to the trouble of raking up past history to show that the redness did not matter; but, if he has enjoyed the vision of redness for a long time, his whole mental equilibrium may

be destroyed if I take it away. An archdeacon has been defined as one who performs archdeaconal functions. Now, sometimes we laugh at that definition; but, provided that the functions are expressed explicitly it is a very good definition—a much better definition for example, than one which defines the individual in question as one who wears gaiters and a top hat of rather ostentatious shape. It is true that the gaiters and the top hat are the most immediately obvious features of the archdeacon, but one who riveted his attention on these appendages as the most fundamental attributes of an archdeacon might be at a loss to understand the significance of the individual if for some reason he were without them, in spite of the fact that he might be just as good an archdeacon. The obvious part of the archdeacon is irrevalent to his true functions. And, so in science, we encounter many instances where we choose to think of things mainly in terms of their activities, and we are loath to add to them, appendages which may hinder of complicate those activities.

THE MID-VICTORIAN AETHER

The scientific thought of the latter half of the last century was concerned greatly with the properties of a hypothetical medium, the aether, whose function it was to transmit light and heat through space, and which later had added to it the duty of carrying wireless waves—a medium in whose subtle properties it was hoped some day to find the secret of gravitation, and everything else which yet held a secret in the universe. Now it turned out to be perfectly possible to express the properties of this medium in mathematical form; but the mathematical equations looked cold and lifeless. One would have liked to think of the aether as something like water—but that would not do, because the kind of waves which a substance like water would propagate are quite different from those which would be required for light. A solid would have been better for the purposes of light. But how would it be possible for the earth and all the other planets to glide so smoothly through a solid. And so there was much puzzling of brains to understand how a medium could on the one hand act like the most limpid of fluids, and on the other hand, like the most rigid of solids. The properties of the required aether were expressed very beautifully, and with perfect consistency by the mathematical equations, but there was no other stuff in the universe which seems anything like what this aether had to be, and many of us did not like it. The aether was like an archdeacon who was perfectly clearly described as regards the duties he performed but who

was otherwise such an unconventional fellow that he could not be made to wear any orthodox top hat, and with legs such that nothing which looked anything like gaiters could be made to adorn them. Finally, when the theory of relativity came, the aether became like an archdeacon to whom even a body was an encumberance, and to whom then was left nothing but a sort of spiritual significance appropriate to a being who performed his ecclesiastical functions without the adornment of head, arms or legs. The mathematician really found little else to talk about in the aether other than its properties as expressed by the mathematical equations, and he became accustomed to dislike the very use of the word aether since he knew that anyone who talked about the aether other than in terms of its purely mathematical properties was really in his subconscious mind thinking of something like water or some other substance familiar in his everyday life. Nevertheless, if we are materialistically inclined, we shall not willingly give up the picture of some sort of medium. We shall view it with the mind's eye, partially shielded by a hazy screen through which we may hope to perceive suggestions of what we hope is reality, but distorted so far that we may never be compelled to pin ourselves down to a statement of what they signify. Thus, while we would strenuously deny that our picture of an all-pervading aether was like that of water, many of us will supplement the cold statement of properties of the medium with vague shadows of substantiality concerning which only the most dire intellectual torments would bring us to confess, even to ourselves, that we were semiconsciously thinking of the taste of the aether, of its boiling point, or freezing point, of its latent heat of evaporation, and so forth.

Lord Salisbury once defined the aether as a word designed to provide a nominative case to the verb "to undulate." I am going to give another definition, which I think I have uttered to some of you before, a definition appropriate to him who is too much engrossed in these shades of reality; and, for his benefit I would say that the aether is a medium invented by man for the purpose of propagating his misconceptions from one place to another. Of all subtle fluids invented for the stimulation of the imagination it is the only one which so far has not been prohibited.

In the historical development of such a theory as the electromagnetic theory, it frequently happens that the ways in which conclusions are reached are founded upon the considerations of special cases, and sometimes are based upon experiments whose representatives have completely evaporated in the more general field. Then, at a later date, in order

to impart logical precision into the developments, we are compelled to take the results which have been stumbled upon in the historical development, review the path by which they have been reached, remove from them as much as possible of the obsolete debris with which it is encumbered, and try to construct some more satisfactory path by which the results might have been attained, even to the extent of possible modifications of the original starting point. This method runs the danger of appearing artificial. Many of the proofs which seemed fundamental to the old structure now play no part at all, and the whole vision may be apt to appear as a constellation in the sky instead of something standing on the rocks of firm earth. However, an honest and unbiased facing of the situation in such a case will usually reveal that, in the last analysis, the older and apparently substantial concepts will not bear transportation to the realms in which the subject claims some of its most powerful uses. Those things which seems to endow the science with such elements of tangibility when considered on an engineering scale dissolve in meaning, leaving only the fundamental content which, as a matter of fact, was the thing really of moment in the engineering problems where, however, it remained hidden in the glittering robes of a spurious reality. And so, to the critically minded, there appears a reality in the new artificiality and an artificiality in the old reality. The process of generalization is often not one in which the new forms a complication, or extension of the old, but rather one in which the old is a rather vague, somewhat incomplete and illogical application of the principles of the new. When as the result of a merciless stripping of irrevalent adornment from the laws, we have a spectacle impalatable to the intellectual taste, let us ask ourselves whether this cold remnant does not contain the whole message of those laws in the sense that they are actually used. When in science we meet things which have few of the attributes of the things around us, when the molecule carries no semblance of color, when the aether has no means whereby to freeze, we must not be alarmed; for the inanimate workers of science carry no insignia of office, and for them of all things it may be truly said "By their fruits alone shall ye know them."

It was early realized that a medium which was to operate according to dynamical principles and serve to produce the most characteristic phenomena of light must be an incompressible medium, a medium which offered no resistance to distortion of its volume element, that is no resistance to shear, but one which resisted mere turning or rotation of those volume element with a couple proportional to the angle of turn.

The forces resulting from such couples combined with the familiar line of crystalization of thought involved in

$$\text{Force} = \text{mass} \times \text{acceleration}$$

glorified into its analytical equivalent, the principle of least action, lead to a system of equations in which the vector measuring the angle of rotation of a volume element was propagated through space with constant velocity and with the property that it lay in the wave front. A similar property held for the vector representing the velocity of the medium.

THE MAXWELL-LORENTZ THEORY

It was McCullagh who discovered the equations defining the properties of the medium, and later Maxwell developed his famous theory of electro-magnetism, in which it appeared that, in free space, away from electric charges or currents, the equations controlling the behavior of the medium were identical with those of McCullagh provided that velocity of the medium be regarded as synonymous with magnetic field and the vector representing the rotation of a volume element be regarded as synonymous with, or proportional to, the electric field. The role of electric and magnetic vectors could be interchanged in this respect in free aether, but considerations following from the more general form of electro-magnetic equations in which electric charges were present pointed to an association of the magnetic field with the velocity of the medium.

However, while a contemplation of the idealistic properties of a pair of unseeable, unfeelable, unsmellable vectors distributed throughout space is stimulating to the æsthetic taste, it is only when the vectors descend to the low estate involved in doing something that they make contact with hard reality. The electro-magnetic field had to move electric charges. The laws according to which it produced such motion were not contained in the field equations themselves, and an appeal to the dynamic picture of a medium with rotational elasticity brought forth no offer of assistance from that quarter. It is true that in the hands of Lorentz, and Larmor, and even in more primitive form in the hands of Maxwell himself, a development of the laws of electro-dynamics from a dynamical status was attempted with more or less success in such a form as to include the laws governing the motion of the electricity itself. However, a careful scouting of the whole procedure leaves one with the conviction that what is really accomplished is a discovery of the quantities which it is appropriate to call kinetic and potential energy per unit

volume of the medium in order that the customary dynamical procedure shall lead to the laws desired. In any case, the procedure lacks that characteristic inherent in the old elastic solid theory, and even in the theory of McCullagh, the characteristic of understandableness from the point of view of mid-Victorian physics. Such comfort as the physicists of a decade ago could have derived from a contemplation of the rotational elastic theory of McCullagh as a picture of the inner mechanism of the electro-magnetic field equation evaporates when the motion of charges must be considered. What significance then remains to the celebrated electromagnetic equations which are here represented?

$$(1)\quad \frac{\rho \boldsymbol{u}}{c} = -\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \operatorname{curl} \boldsymbol{H}, \qquad (3)\quad 0 = \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} + \operatorname{curl} \boldsymbol{E}$$

$$(2)\quad \rho = \operatorname{div} \boldsymbol{E} \qquad (4)\quad 0 = \operatorname{div} \boldsymbol{H}$$

It is possible to solve these equations in such a manner as to express the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ in terms of the distribution and motion of the electricity of the system. Those who are familiar with the subject will recall that these solutions take the form:

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{U}}{\partial t} - \operatorname{grad}. \phi \tag{7}$$

$$\boldsymbol{H} = \operatorname{curl} \boldsymbol{U} \tag{8}$$

where

$$4\pi\phi = \int\int\int \frac{[\rho]}{R} d\tau \tag{9}$$

$$4\pi c\boldsymbol{U} = \int\int\int \frac{[\rho \boldsymbol{u}]}{R} d\tau \tag{10}$$

ϕ and $\boldsymbol{U}$ being restricted by the relation

$$\operatorname{div} \boldsymbol{U} = -\frac{1}{c}\frac{\partial \phi}{\partial t}. \tag{11}$$

It is of course understood that the square brackets here used are to be taken as indicating that the potentials are calculated in the usual retarded sense. All that I wish to say about these equations is this. They tell us that when we are given the density of the electricity at every point of space and the velocity at the present and previous times we are in a position to calculate the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$. I prefer to speak of the process of calculation of these vectors as definitions of the vectors,

and to regard that process as containing no statement of law drawn from nature herself. But you may well object, that it is impossible to discover laws of physics by definitions, and you may question the value to be attached to an apparently arbitrary set of definitions of this kind. The point where nature steps in is in providing for the fact that the quantities which we have defined in this rather complicated way shall be of some use. In classical electro-dynamics it was in terms of these quantities that the motions of charges could be expressed with more or less simplicity by an equation which for our present purposes it will be sufficient to take as

$$m\ddot{S} = e\left(E + \frac{[vH]}{c}\right). \tag{5}$$

The field equations governing E and H may be regarded as an intermediate scaffolding convenient for the purpose of expressing in artistic mathematical form what would otherwise be a very complicated relation between the motion of any one electron and that of all of the others. It is true that we cannot discover laws of physics by definition; but, we can endow quantities with useful and suggestive properties by definition. The quantities E and H defined by the electro-magnetic equation possesses, as I have already remarked, the important property associated with transverse waves, the property that the vectors E and H lie in the wave front. They are, therefore, suitable quantities in terms of which to talk of the ultimate realizable thing in light whatever it may be which lead the experimentalist, early in the history of the subject, to the knowledge that there were certain pecularities about light as distinct from such a phenomenon as sound, pecularities which in terms of a picture of a wave motion propagated through space called for transverse waves in the case of light vibrations. Today we have found it necessary in atomic phenomena to discard that simple law represented by equation (5) for the motion of an electron in terms of the vectors E and H. However, the properties planted in E and H through the electromagnetic equations have caused them still to retain their place as the fundamentals in terms of which the description of optical phenomena may be made.

In speaking of the motion of one group of electrons in relation to the positions and motions of another group, we may draw an analogy between two buildings in which there is obviously some relation between the forms of structure. It might be possible to put up a scaffolding between these two buildings and to express the relation of their forms of

structure, which in themselves might appear rather official and complicated, in terms of a relatively simple structure concerned with the scaffolding. Our electro-magnetic equations are analogous to the scaffolding. The scaffolding itself was put up by the builders. It need not even exist in the final structure at all, but nevertheless its introduction even in the mind's eye might serve as a convenient help for describing the relation between the two buildings. So the reality or non-reality of the vectors associated with the electro-magnetic equations, whatever that term "reality" may mean, is not a thing which need concern us very much, so long as those vectors themselves, real or unreal, serve as a means of correlating the things which we do in the act of producing light, with the things which we finally observe; or speaking more specifically, provided that they serve to correlate the motions of one group of electrons with those of another. Having said this it is of course a convenience to stimulate our imagination by thinking of those vectors $\boldsymbol{E}$ and $\boldsymbol{H}$, or at any rate of a wave motion, because many of the characteristics which it is of interest to discuss in light depend upon the phase relations between the different contribution to those vectors arising from different origins; and, for many purposes, the phase relations may be discussed just as well in terms of water waves, in which we have done our thinking in childhood, as in terms of the more abstract quantities of electro-dynamics.

EARLY QUANTUM THEORIES

The Maxwell-Lorentz system of electro-dynamics was so successful in co-ordinating the broad features of optics, and gave such promise in respect of some of the more fine grained phenomena such as the Zeeman effect, the theory of normal and anomalous dispersion, etc., as to suggest that only a more detailed knowledge of the internal mechanism of the atom was necessary in order to complete the harmony of the situation, and provide for a rounded out scheme of optical phenomena. However, even towards the end of the last century, two sets of phenomena stood out to challenge any contentment on our part that the final solution had been reached. In the first place, there was the law of black body radiation which seemed to deny all that might have been expected of it from the conventional line of thought of the day. Then, more intimately related to our subject, was the matter of spectral series, of the laws governing the wave lengths of the different spectrum lines which are emitted by any incandescent gas. Every attempt to correlate these laws with the customary usages of classical dynamics and electrody-

namics, pointed to fundamental differences between what was found and what was expected. Then came that great revolution in our picture of these matters introduced by Niels Bohr.

Those who have grown old in building the frame work of thought in which so much of nature has been revealed, hesitate to tamper with that frame. Bold and even radical in their youth, they become conservative with age. And, even as a great statesman hesitates to tamper with the constitution which he himself has helped to build, fearful lest the slightest change may bring dire results, so even the giants of science become timid in speculation, and it is often to genius in its youth that the world must turn to unravel the harmonies of the new composition, and mould them with each other and with the past, so that the whole may not stand as a bizarre and formless mass of facts, but as a new symphony in the realm of all nature's beauties. And so we find Bohr introducing several strange and radical ideas. The atom is a planetary system; but the electrons which revolve about its nucleus secure from some unknown potentate an indulgence whereby they may refrain from radiating energy while travelling in their orbits. However, when the electron is removed from one orbit to some other farther from the nucleus it possesses the power to return to that orbit, cancelling all obligations as regards its previous residence, and assume a new purely ornamental position by the simple expedient of radiating the energy available by passage between two orbits, in the form of a monochromatic frequency of amount equal to the energy change divided by Planck's constant h. The law which decided what orbits were to be possible in the atom was artificial in nature but simple in form, and we all know how this picture lent itself to a very remarkable correlation with the spectrum of hydrogen. Moreover, the straining of the dynamical problem to its limit in taking account for example of the slight motion of the nucleus during the electron's orbital motion, served only to improve the picture and provide for the slight difference between the ratio 4 to 1, and that actually found, for the ratio of the fundamental constants of the spectrum of ionized helium and hydrogen. The extension of this theory in the hands of Sommerfeld, served to reveal in most beautiful manner the details of the fine structure of the hydrogen spectrum. For the relative intensities of spectral lines, Bohr found it necessary to invent his well-known correspondence principle, which, in spite of its ingenuity in conception and fruitfulness in prediction in the more elementary cases, always suffered from the philosophical objection that it sought to account for what would happen in a certain case by setting up a correla-

tion with what we should get in another case if something happened which does not happen. However, it was not so much objections of this kind which mattered, as the fact that, in the more complicated atoms, at any rate, the very formulation of the correspondence principle became ambiguous, and even the fundamental quantum conditions themselves lost their analytical identity. This particular field, however, is not one which is pertinent to my present address. The thing which is of significance to us is that the Bohr theory left the emission of light itself an untold story. In the transition from one energy state to another the light was emitted in the form of a constant frequency given by the change of energy divided by h and that was the end of the matter. We might have been content without an æther picture of the light vibrations, but in our conventional state of mind, we at least want to be able to trace the vibrations of a spherical wave for example, inwards towards its center of origin, and have some sort of a look at the thing which produces them even though we may never have seen that animal before. The most that was attempted in this direction in the form of a superstructure to the Bohr theory was the introduction of the conception of virtual oscillators. Having witnessed in our minds a quantum transition in the atom we pocketed the energy, suddenly changed our picture to that of a simple harmonic oscillator of the oldest vintage, presented the energy to it, and allowed it to do its worst. In problems of absorption, of the reflection and refraction of light, and of all of those phenomena in which the old classical theory had worked so well with real oscillators, we now invoked the picture of these virtual oscillators, replacing our older conceptions of real oscillators actually present with any given frequency by virtual oscillators whose number for the purpose in hand was determined by the probability of transition in the atom associated with the corresponding frequency. Even then, however, our troubles were not over, for these virtual oscillators at least forced us to the picture of an ordinary electromagnetic wave. Several phenomena, such as the photoelectric effect, the properties of x-rays, and so forth, presented difficulties in respect to this picture which increased continually the more we learned concerning the phenomena themselves. The natural picture is that of a continuous wave front falling upon a metal, for example, and acting towards each electron which it encounters in the same way. Yet the facts suggested that the photoelectric emission of an electron from an atom was such a rare occurrence that it would have to be classed in the category of the most wonderful of miracles to any being

who might be supposed to live on the atom. Attempts were made to avoid the difficulty by assuming that the photoelectric effect was a phenomenon peculiar to certain atoms which were in an exceptional state. This idea was strained to its limit to provide a satisfactory explanation, but it always stood out as an artificial assumption suggested by no other consideration of different kind, and one out of keeping with all that might be expected of atomic behavior. The study of x-rays served only to widen the gap between the predictions of the classical mechanism and the observed facts.

Some of the simplest and most characteristic phenomena in optics and x-rays found their explanation in the idea that light and x-rays were emitted in the form of particles of very small size carrying with them an amount of energy $h\nu$ where h is Planck's constant and ν the frequency, and an amount of momentum equal to $h\nu/c$. It was perhaps in the Compton effect that this picture showed its strongest potentiality. As you will recall, if x-rays be allowed to fall upon an atom, the rays which are scattered suffer in part a change in frequency in the direction of the longer wave lengths. For each scattered x-ray, there is an ejected electron; and, on the basis of the assumption that the x-rays are particles whose energy is equal to h times their frequency, and whose momentum is equal to the energy divided by the velocity of light, and by writing down an equation which expresses the conservation of energy and momentum as between the original ray on the one hand, and the scattered x-ray and the electron on the other, it is possible to give a satisfactory explanation of the relation between the direction of the original x-ray, and that of the scattered electron and the scattered x-ray; and, it is also possible to account for the relation between the energy of the ejected electron and the alteration of wave length of the x-ray.

The picture of particles, however, was about as ill-adapted as anything could possibly be to an explanation of one of the most outstanding characteristics of light known from the time of Newton, the characteristic exhibited in interference phenomena. Attempts at a compromise between the particle idea and that of wave motion through the supposition that light was of a filamentary nature propagated along isolated aether threads were made long ago, as soon in fact as the difficulty suggested by the photo-electric effect and by x-rays suggested themselves. However, the filament hypothesis refused to give forth anything more than was put into it, and indeed refused to accept into itself much that was required. On the one hand, for the purposes of the photo-electric effect and allied phenomena it was necessary to assume that the light

filament had cross sectional dimensions comparable with those of an atom. On the other hand, a filament of light which was to retain any of the characteristics suggested by a wave nature of light and of the electro-magnetic theory in particular would, in order to account for the known resolving powers of large telescopes, have to be a thing comparable in diameter with the diameter of our largest telescopes. Again, the fact that interference of light could be obtained over distances comparable with a meter required that the filamentary quantum of light should have a length of at least this order of magnitude, so that from these aspects the light quantum would have to be a thing comparable in volume with a cubic meter in contrast to the demands of the photo-electric effect and x-ray for a quantum almost infinitesimally small. An attempt to bridge these difficulties by supposing that the light quantum consisted of a group of filaments, was reasonably successful as regards terrestrial interference phenomena, but failed to account for interference of light from the stars unless the filaments associated with an individual quantum are so numerous as to deprive the idea of all practical usefulness as regards a concentration of a large amount of energy in any one of them. Then, firm in its claim for attention on behalf of the old classical theory stood out the fact that the relative intensities of light in interference and diffraction phenomena were calculable with an almost uncanny accuracy in terms of the continuous wave theory of light, in terms for example of the electro-magnetic theory.

The situation is this: If on the wave theory of light, we calculate the path along which energy may be regarded as flowing, then the places which are predicted as regions of darkness, the places to which no wave energy is found to travel will, in experimental practice, be found to be regions of actual darkness. All wave theories of light are capable of tracing these regions of darkness, and even as regards polarization phenomena we do not need the full content of the electromagnetic theory to provide for satisfactory correlation with the facts. The essentials of the story are told in terms of the phase relationships of the various elements of vibration which have travelled from the light source to the point under examination by different paths. In the electromagnetic theory, Poynting showed how to calculate the paths of the electromagnetic energy in terms of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$; and, the Poynting flux as it is called, represents, at any point, the flow of electromagnetic energy per square centimeter per second. If we draw the lines of the Poynting flux, then, in space, far removed from electrons, a tube formed with such lines as a boundary possesses the important

property that while its cross-section may vary from point to point, the density of the flux changes in corresponding manner so as to insure that the energy-flow per second through any one cross-section of the tube at one place is equal to the energy flow per second through any other cross-section at another place. If then we are to hold to the principle of light quanta in the form of small particles carrying energy and momentum, we may take it not as a hypothesis, but as a fact demanded by the experiments themselves, that these light quanta must be guided by the electromagnetic field so that they move in such a manner that they never go to places where the electromagnetic theory or for that matter, any other wave theory, would predict darkness. In other words, the light quanta follow the lines of the Poynting flux. In order to crystallize the idea into a form quantitatively consistent with the facts, let us draw a little closed surface surrounding the point from which the light is supposed to come. Here we must necessarily be a little vague since the nature of the oscillator with which we associate the electromagnetic radiation is not understood. We suppose however, that during an atomic transition a quantum of the appropriate energy is emitted with an energy $h\nu$. We shall suppose that the probability dp that it is emitted through an element ds of the surrounding surface in the time dt, is given by

$$dp = APdsdt$$

where P is the magnitude of the Poynting flux, that is to say, the energy propagated per square centimeter per second according to the electromagnetic theory, and A is a constant so chosen that the integral of the right hand side taken all over the enclosing surface and from $t=0$ to $t=$infinity, is equal to unity, so as to insure that sometime or other one quantum shall be emitted as a result of that transition which has taken place. Let us then suppose that the quantum, having been emitted, follows the lines of the Poynting flux. It will of course follow from the assumption made, that on the average, the number of quanta passing through any square centimeter will be proportional, when integrated over a long time, to the energy which would have been calculated as emitted through that square centimeter on the electromagnetic theory. It is important, of course, to realize that we are not postulating two kinds of energy. On the quantum idea, the electromagnetic field as a carrier of energy has evaporated, and the electromagnetic vectors and the Poynting flux associated with them are now to be regarded merely as a kind of scaffolding for guiding the quanta. The whole procedure constitutes, in fact, a method of describing how quanta move from light

sources, how they are reflected from reflecting surfaces, and so forth. The production of interference, for example, by a pair of Lloyd mirrors, is now to be explained by the fact that no quanta succeed in getting to the places where the wave theory predicted darkness because there is no path which follows the Poynting flux and leads to that particular point. In its passage through prisms and lenses, the Poyntning flux becomes, of course, very complicated in nature, but it knows where it's going, and when it gets out on the other side its story as regards the motion of the quanta becomes once more simple. The situation is similar to that in the old wave theory of light, where the path of the light in the atomically heterogeneous refracting medium assumes a very complicated form but one which for many purposes may be smoothed out in a manner concerning which no more need be said about the process than that which is involved in the statement that the velocity in the refracting is constant everywhere but different from what it is in the air. In the case of a number of atomic transitions taking place at about the same time, it is of course necessary to assume either that the chance of two transitions coming together is so small that we need never have occasion to consider the over-lapping of the Poynting flux from the difference sources, or to assume what seems more natural, that in speaking of a quantum being guided by the Poynting flux, we refer only to the Poynting flux associated with the original transition which gave rise to that quantum, and the various secondary Poynting fluxes to which this gives rise by the ordinary processes of reflection and refraction incidental to the production of the interference pattern. From the point of view of the electromagnetic theory, the picture presented by this idea is rather interesting, particularly in relation to the mechanism of the absorption of a quantum. We know that if an atom possesses a resonant frequency equal to that of the oncoming light, the probability of its receiving a quantum of that light is greatly increased thereby. Now, from the point of view of the electromagnetic theory, the existence of such a resonance frequency results in the creation of an electromagnetic field which combines with the field of the oncoming wave in such a manner that the resulting Poynting flux is, as it were, sucked into the atom from a wide solid angle. This feature of resonance thus acts to enhance the probability that the atom in question receives the quantum of the light emitted in the primary transition.

There is, however, one situation, whose mathematical interpretation leads to a rather alarming and some will say, even fantastic concept. I have remarked that the Poynting flux integrated over the cross-sec-

tion of a tube is constant along that tube so long as we concern ourselves with free space. When we come into the vicinity of an atom or a place where electrons are to be found, however, the situation is different. An appreciable part of the electromagnetic energy of the electrons now exist in the portion of the tube between the transverse partitions, for example, at A and B, Fig. 1, and, the integrated flux through the surface A where the flux enters the tube is now less than the integrated flux through the surface B, by which it leaves, by an amount which is

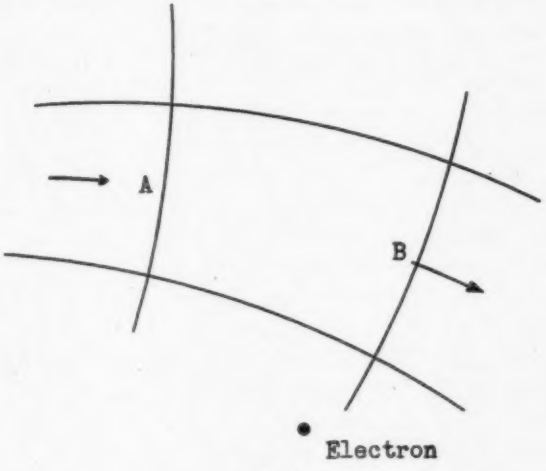

FIG. 1

equal to the rate of increase of electromagnetic energy in the space between A and B. This result is always true, but in free space the rate of increase of electromagnetic energy in question is zero. If now we are to hold to the principle that the Poynting flux determines the probability of passage of a quantum through an element of area perpendicular to it, we must suppose that the probability that the quantum shall pass through the cross-section at B is less than the probability that it shall pass through the cross-section at A in any element of time by an amount proportional to the decrease of electromagnetic energy between A and B in that time. The element of physical interest now arises when we ask the question as to what becomes of a quantum which enters the tube AB at A, and fails to leave it at B. In order to provide for consistency we must suppose that this quantum disappears from the region in question and instantaneously appears somewhere in the atom, it is not in-

cumbent upon us to say where, nor indeed to say more than that it suddenly becomes endowed with something that it has been looking for all its life, the opportunity to do something. It becomes endowed with the power to promote a transition in the atom, for example, or to participate in the production of the Compton effect, or to cause the ejection of a photoelectron. Of course, from a practical standpoint, this sudden disappearance of the quantum from one place accompanied by appearance at another, the solution of the quantum in the aether, if you will, accompanied by its precipitation at another point in space, is not so drastic a matter as may appear at first sight. For, it is only in the immediate vicinity of an atom that the elements of volume of space experience an appreciable rate of increase of energy in the electromagnetic sense as the result of the impact of the oncoming wave. In principle, however, the phenomenon is rather a startling one. Thus, for example, if a light wave coming from the sun acts upon an electron in an atom of the earth, a very small amount of the kinetic energy of that electron whose home is the earth, will reside in elements of volume around the sun. Therefore, even in the case of the Poynting flux starting out from the sun, it is not absolutely true that the flux through areas such as A and B in the vicinity of the sun, are quite equal to each other, so that there is a finite but of course, enormously small probability, that the quantum after leaving the sun, will make a sudden jump and appear in the atom of the earth. Or, if you will, you may think of it as dissolving in the aether at the sun and being precipitated at the place where its work is to be done in the atom on earth. One further thing we must refer to. In spite of the fact that the quantum may be bent very considerably in its path as it approaches the immediate vicinity of an atom, we must suppose that it conscientiously remembers everything that has happened to it, and that when the final catastrophe occurs, it operates with the momentum and energy which it had imparted to it at the instant when it left the region of the primary transition. There is of course, no difficulty in formulating in consistent manner all of these requirements. Some of them, however, rather grate upon that particular region of sensitivity in our intellectual makeup, the region where resides what we call physical intuition. But, insofar as physical intuition is the extrapolation of our previous experience, we must be chary of allowing it to become our master in those cases where we are discussing phenomena whose very interest lies in the fact that they are outside of our realm of former experience.

In such a view as that which I have discussed, the production of such

a phenomenon as the Compton effect, would, by the initian of a new transition, give rise to a new electromagnetic radiation, which would determine the probability of emission of a new quantum with energy and momentum characteristic of the frequency of that radiation. This new electromagnetic field would then operate independently of the primary field, and in connection with its reflections and refractions, would provide for a Poynting flux to tell the story of the history of the new quantum.

THE MATRIX THEORY

As the difficulties of the Bohr-Sommerfeld theory piled up, it became evident that a theory which was to tell in consistent manner the story of the laws of spectral series, and give at the same time the relative intensities of spectral lines must assume a form radically different from what we already had. And so, there arose the Matrix theory of Born, Heisenberg, and Jordan. It is not my purpose to discuss this theory in detail. I must content myself by reminding you that, starting with a certain mathematical form, the Hamiltonian function, which figured prominently in the Bohr-Sommerfeld theory of the atom, it set up a new system of rules, or mathematical operations, by which, in terms of this form one could define a set of simple harmonic oscillators whose frequencies were, in the case of the hydrogen atom at any rate, the same as those required by the Bohr theory. The amplitudes of these oscillators was then taken as a basis in terms of which to calculate the intensities of the various frequencies emitted. The procedure at this point assumes a form artificial and formal to the highest degree. Insofar as the mathematical structure suggests a picture at all, the picture is that of a number of simple harmonic oscillators vibrating all at the same time and continuously forever, without indeed the emission of any radiation, since those elements which provide for dissipation of energy in the classical theory are absent in the matrix theory. However, in the physical interpretation of the results secured, a return is made to the idea of an atom passing from one state to another and emitting during that transition a definite amount of energy $h\nu$, and it is in defining a probability to this transition that the amplitude in the corresponding frequency in the matrix problem is invoked, I must not say to tell the story, but to set up a formal procedure for calculation. The frequency in the matrix theory is given by the difference of two numbers characteristic of the atom, divided by Planck's constant h. These numbers are associated with the energy states of the atom. Attention is fixed on one of them,

and upon all the frequencies which may be produced by a transition from that value to a lower value. The rate of emission of energy by a matrix theory oscillator of one of these frequencies is calculated according to ordinary electromagnetic principles. This rate divided by h gives the number of quanta which would have to be emitted per second, if on the average the energy emission in quanta were equal to that predicted by the electromagnetic theory. The values of the quantities so obtained for the different frequencies which may be considered as arising from transitions with a common upper energy level, are then taken as proportional to the probabilities of the corresponding transitions. Of course, the matrix theory attempts no answer to the question as to the relative probabilities of the different stationary states from which the transitions occur. Indeed, it dare not do this. If it succeeded it would do too much, because the relative probabilities of the different stationary states is a matter for the experimenter to determine. In the normal hydrogen atom, the electron may be supposed to exist in its lower energy state. By suitable excitation it may be raised to any of the other states, but the state to which it is raised, will of course depend upon the experimental procedure adopted.

Thus, as regards the emission of the light itself, the matrix theory adds nothing to the old Bohr theory. The situation is simply left in the condition that, in a transition involving an energy change E, a light frequency is emitted of amount E/h, and that is the end of the story, apart from the expressions of the probability of emission. Nothing is said as to whether it is emitted in the form of a wave, a filament, or a corpuscle, and all of those difficulties inherent in the demand for a corpuscular aspect for such phenomena as the Compton effect and the photoelectric effect on one hand, and the demand for a spreading wave to account for interference on the other, remain as before.

Wave Mechanics

Hard upon the heels of the matrix theory came the theory of Schrödinger. Now the significance and power of this theory and of its subsequent development is largely a function of the manner in which one uses it. Some of the features upon which stress was laid in the earlier developments seem of relatively small importance, and I shall say little about them. You will recall that in the classical dynamical statement of mechanical problems, the Hamiltonian equations played a fundamental role. They are represented in equations (6) and (7).

$$(6) \quad \dot{p}_r = -\frac{\partial H}{\partial q_r}; \quad \dot{q}_r = \frac{\partial H}{\partial p_r} \tag{7}$$

H is the Hamiltonian function which, in general, is a function of let us say n momenta p, and n coordinates q, and possibly also of the time, although for conservative systems and indeed for much that it is necessary for us to talk about now, it will be sufficient to omit this dependence upon the time. The old Bohr theory presupposed a model, and once a model had been assigned, it was possible to write down the Hamiltonian function and proceed, at least to a solution of the equations of motion. In the Schrödinger theory we seize upon this Hamiltonian form as a starting point. We take it as something characteristic of the atom, but we no longer think of it in its old dynamical role. We rub out the momenta p_r, and replace each of them by the corresponding quantity $h/2\pi i\ \partial/\partial q_r$. In this way we construct in place of the Hamiltonian function, a Hamiltonian operator. Using this operator as a basis we write down the differential equation

$$-\epsilon\frac{\partial \psi}{\partial t} = H\left(q, q, \cdots, \epsilon\frac{\partial}{\partial q_1}, \epsilon\frac{\partial}{\partial q_2} \cdots t\right)\psi \tag{8}$$

where $\epsilon = h/2\pi i$.

We seek a solution of this equation of the form

$$\psi = uT$$

where u is a function of the coordinates, and T is a function of t. This leads immediately to

$$\frac{-\epsilon}{T}\frac{\partial T}{\partial t} = \frac{1}{u}H(u).$$

The left hand side is a function of t only, and the right hand side is a function of the q_r only. We consequently equate each of these to a constant E, which, so far, may have any value, and so we obtained the two equations

$$\frac{-\epsilon}{T}\frac{\partial T}{\partial t} = E \tag{9}$$

$$\frac{1}{u}H(u) = E. \tag{10}$$

The solution of the first of these equations is

$$T = \text{const} \times e^{-2\pi i E/h t} \tag{11}$$

so that we may say

$$\psi = ue^{-2\pi iE/h} \tag{12}$$

where u is a solution of (10).

Now, the interesting thing is that equation (10) has, in general, solutions for u which vanish at infinity and which otherwise are finite, continuous and single-valued throughout all space, only for certain values of E; and, for example, if the H we originally started with is the H of the hydrogen atom, then it turns out that these values of E for which this equation has the kind of solutions in question are exactly those which we calculated as the possible values of the energy according to the old Bohr-Sommerfeld theory. Here then is a suggestive fact in terms of which to try to build up something. For the moment we confine our attention to those solutions of (10) and the corresponding values of E which do correspond to solutions finite and continuous throughout space, and vanishing at infinity. And so with these as a basis, we construct a solution of the equation (8) of the form

$$\psi = a_1\psi_1 + a_2\psi_2 + \cdots a_m\psi_m \cdots \tag{13}$$

where $\psi_m = u_m e^{2\pi i E_m t/h}$, and where the a's are real constants.
The values of the u_m, which are solutions of (10) are, of course, ambiguous to the extent of an arbitrary constant. We adjust the constant in each case so that[1], with the integrals taken throughout space,

$$\int\int \cdots \int\int u_m^2 dq_1 dq_2 \cdots = 1 \tag{14}$$

and it is a point of interest that for all values of m not equal to n

[1] There is nothing mysterious in the adjustment of the multiplicative constants to which I have referred. The situation is exactly analogous to what we should have if we had the differential equation

$$\frac{d^2y}{d\theta^2} + y = 0.$$

If we had not yet discovered the Sine function we should find as a solution for this equation

$$y = a\left(\theta - \frac{\theta^3}{3!}\cdots\right).$$

The constant a would be arbitrary, and if we had the good sense to choose it equal to unity, we should have many interesting properties about this supposedly newly discovered function, such properties, for example, as would insure that

$$\int_0^{\pi} \sin^2\theta d\theta = \pi.$$

$$\int\int\cdots\int\int u_m u_n dq_1 dq_2\cdots = 0 \tag{15}$$

at least for the case where the derivatives with respect to the q's in (8) combine to the usual form $(\partial^2/\partial q_1^2+\partial^2/\partial q_2^2+\partial^2/\partial q_3^2+\cdots)$.

We now form the conjugate of ψ as given by (13). It is obtained by simply replacing i by $-i$. Calling it $\bar{\psi}$, we form the quantity $\psi\bar{\psi}$, which is

$$\psi\bar{\psi} = a_1^2\psi_1\bar{\psi}_1 + a_2^2\psi_2\bar{\psi}_2 + \cdots + a_1a_2(\psi_1\bar{\psi}_2 + \bar{\psi}_1\psi_2) + \cdots + a_m a_n(\bar{\psi}_m\psi_n\,\bar{\psi}_m\psi_n) + \cdots .$$

Writing $u_m = |u_m|e^{2\pi i\delta_m}$
where $|u_m|$ is real, we readily find

$$\psi\bar{\psi} = a_1^2|u_1|^2 + a_2^2|u_2|^2\cdots 2a_m a_n\sum\sum|u_m||u_n|\cos 2\pi\left\{\frac{(E_m-E_n)t}{h} + (\delta_m-\delta_n)\right\}\cdots . \tag{16}$$

It will be observed that the quantity so defined consists of a number of simple harmonic frequencies of the form $\nu_{mn}=(E_m-E_n)/h$, where the E's are the characteristic constants of the atom, the quantities which, in the old Bohr theory, would have been the energy levels.[2]

Now it results from (15) that the quantity $\psi\bar{\psi}$ as given by (16) has the property that when integrated throughout the whole of the multi-dimensional space of the q_r it gives a result independent of the time. In a three dimensional case, such as corresponds to a single electron, this result reminds us of the constancy of the total charge, so Schrödinger takes the quantity $\psi\bar{\psi}$ as representative of the charge density, for a three dimensional problem.[3] For this case he creates an electric moment out of it by multiplying the value of $\psi\bar{\psi}$ in any element of volume by, let us say, the x component of the element of volume and then integrating the result throughout space. The result when multiplied by the electronic charge is associated with the x component of the electric moment. It, of course, varies periodically with the time with the same frequencies as those with which $\psi\bar{\psi}$ itself varies.

[2] In order to avoid complexities which would play no important part in the present article, I have not dissected the E_m and labeled them according to their quantum numbers. In the above, each individual value of the E's, is supposed included.

[3] The actual magnitude of the charge is simply a question of the units; and, we can incorporate with (16) a suitable multiplicative constant to make it anything we wish.

In the case where there are many degrees of freedom, as in the case of an equation of the type (8) which has arisen from a Hamiltonian function for an atom containing many electrons, the electric moment of the system is taken (in the case of the x component, for example), as

$$Q = \sum_r e_r \int\int\int \cdots \int\int\int (q_r)_x \psi\bar{\psi} dq_1 dq_2 \cdots \qquad (17)$$

where the q's occur in groups of three for each electron; and $(q_r)_x$ refers to the coordinate which is associated with the x component of the r^{th} electron—associated in the sense that the r^{th} electron determined the occurrence of that coordinate in (8), although in that equation it, of course, may partake of all values, and has no reference to the actual numerical value of a coordinate of any electron. The summation in (17) is taken for all the electrons which originally participated in determining the Hamiltonian function. The quantity Q also varies periodically with the time with the frequencies determined by (16); and, as a matter of fact, it is convenient to regard it as the sum of a number of parts

$$Q = \sum_m \sum_n a_m a_n \left\{ \sum_r e_r \int\int \cdots \int\int (q_r)_x \psi_m \bar{\psi}_n dq_1 dq_2 \cdots \right\}. \qquad (18)$$

The m, n, term of this sum is associated with the frequency $(E_m - E_n)/h$. Schrödinger then considers the various contributions to the moment Q associated with the various frequencies as representative of simple harmonic oscillators; and the remainder of the story as regards the radiation emitted by these oscillators is supposed to go on according to the ordinary electromagnetic theory in exactly the same manner as was the case in the matrix theory.

Now one may well ask "what is the justification for this procedure?" Perhaps the most powerful answer is to say that, in spite of the drastic difference between the analytical procedure in this case and in that of the matrix theory, it can be shown that the two are equivalent in the sense that the simple harmonic oscillators arrived at in this way have exactly the same frequencies and amplitudes as those arrived at in the matrix theory, provided that the starting point in both cases is the same Hamiltonian function. In other words, the contribution from (18) which represents[4] the instantaneous x component moment of frequency $(E_m - E_n)/h$, i.e. the contribution

[4] Regarded here in its complex exponential form for convenient compression with the matrix theory.

$$Q_{mn} = a_m a_n \sum_r e_r \int\int \cdots \int\int\int (q_r)_x \psi_m \bar{\psi}_n dq_1 dq_2 \cdots \qquad (19)$$

is numerically equal to the instantaneous moment as calculated from the matrix theory for the corresponding frequency. Incidentally, the quantity which, in the Schrödinger theory plays exactly the same role as the total x component momentum associated with the frequency ν_{mn} of the oscillator in the matrix theory, is[5] proportional to

$$P_{mn} = \frac{h}{2\pi i} a_m a_n \sum_r e_r \int\int\int \cdots \psi_m \frac{\partial \psi_n}{\partial (q_r)_x} dq_1 dq_2 \cdots . \qquad (20)$$

Although perhaps not consistent with the historical method of development, we may say that the realization of this correspondence is a justification for fixing the arbitrary constant in the u_m's by equation (14). Moreover, the decision to limit ourselves to solutions of equation (10) which are finite, continuous, and single valued throughout all space and vanish at infinity assumes a less artificial form, at least for the three dimensional case, when we realize that were we to include other solutions, solutions which were finite or infinite at infinity for example, they would correspond to finite or infinite charge densities at infinity. The restriction, therefore, to these particular solutions is simply the analytical equivalent of confining our interest to those problems in which the charge density is zero at infinity.

In the case of an atom with many electrons, where there is a correspondingly large number of degrees of freedom, the integration which, for the three dimensional space extended simply over the whole of that space, extends now over the whole of the multidimensional space, and the physical meaning of the charge density loses much of its reality. However, in spite of the artificiality, of the process, we do arrive at a set of definite numbers for the atom, and numbers which vary with the time periodically, and with the frequencies desired; and these quantities can be taken as sorts of virtual oscillators in terms of which to discuss the radiation emitted. However, I do wish, even at this stage, to remark upon one feature which is involved in the process, and which seems of doubtful validity in its general application. Strictly speaking, in view of the finite distance of the various volume elements from each other, a

[5] The q_{mn} and p_{mn} appearing in the matrix theory represent the quantities given in (19) and (20) but without the summations and without the factors e_r and $a_m a_n$. The latter factor is associated with the probabilities of the different energy states. To obtain the total effectve moment corresponding to any state of polarization, we must of course, use the quantity Q_{mn} given above.

consistent treatment which regarded $\psi\bar{\psi}$ as a charge density, would not attempt to lump all of the electricity together into one single moment, but would consider the contributions of each element of volume to the electromagnetic field to be produced, taking into account the relative retardation between the different elements of volume as the result of their different positions. The situation in this respect is probably of minor importance in the case of spectral emission since the dimensions to which the charge is sensibly confined, are small compared with the wave length. There are conditions, however, in which the lumping of the charge together in the form of a moment might have very serious consequences. Thus, for example, thinking, for simplicity of a three dimensional case, suppose that we should have, as indeed in some cases we do have, a situation in which the ultimate $\psi\bar{\psi}$ function is of such a form that its phase is constant over planes parallel to some fixed plane, but of course, with a value depending upon the distance from that fixed plane. It is obvious that the electromagnetic radiation emitted from such a system, will be emitted unidirectionally. It will not spread, but will travel as a parallel beam in a direction perpendicular to the plane. If, however, we should attempt to lump the whole of the $\psi\bar{\psi}$ together into a moment, which operated from a single point, then it is obvious that we should obtain a spherical wave characteristic of a dipole oscillator. The situation is analogous to that which we have in a parallel beam of light produced, for example, by a point source at the focus of a lens. Across a plane A perpendicular to the parallel beam we have a condition in which the vibrations are all of the same phase. I can express the intensity of the radiation at any point on a plane B ahead of the plane A, in terms of the situation at A; and, if I do this, I of course, find that it is zero outside of the beam, and is constant inside. If, however, I should attempt to combine into a single moment, operative at point, the whole of the disturbance operative over the cross section of the beam at A, the best that I could hope for would be a spherical light wave of the type associated with a simple harmonic oscillator.

The quantity $\psi\bar{\psi}$ having been associated with charge density, it of course became necessary to pursue this idea to its logical conclusion. The picture of an electron travelling through space was to be represented by a little wave packet in $\psi\bar{\psi}$ travelling also through space. It was hoped that the little wave packet would succeed in holding itself together; but, in spite of the encouraging promise in this respect in the matter of simpler cases, it soon appeared that such a condition would not hold. The Almighty had honored many of the checks written on this

theory so far, but he failed to cash that one. And now of course, there was a pretty mess, because even the electron would presumably fail to hold together during its wanderings. The next step was to give up the direct association of charge density with $\psi\bar{\psi}$ and regard the latter quantity simply as a measure of the probability of the presence of the electron in the particular element of the space to which it was attached.

The starting point is the hypothesis that the quantity $\psi\bar{\psi}$, given as a solution of (8) in the form of a function of the q's and t, is to be a measure of probability in the sense that the probability that electron no. 1 lies in the range q_1 to q_1+dq_1, that electron no. 2 lies in the range q_2 to q_2+dq_2, etc., is dP, where[6]

$$dP = \frac{dq_1 dq_2 \cdots dq_n \psi\bar{\psi}(q_1, q_2 \cdots t).}{\int\int \cdots \int\int \psi\bar{\psi} dq_1 dq_2 \cdots dq_n} \tag{21}$$

and where the integral is taken over the whole range of the multi-dimensional space, and the constants a contained in it are restricted so as to make it equal to unity, so that

$$dP = dq_1 dq_2 \cdots dq_n \psi\bar{\psi}(q_1, q_2, \cdots t). \tag{22}$$

Now it will be well to do at the outset something that is not usually done, but which might as well be done, viz: to remove some of the abstractness from the $3n$ dimensional space. In a sense, the $3n$ dimensional space is to be regarded as n three dimensional spaces which are all really the same old three dimensional space of our experience considered separately for each electron. The non-mathematical reader may like to think of them as all superposed, and kept distinct in the mind's eye by drawing the axes, elements of volume, etc., of one in red ink those of another in blue ink, and so on. Each space is for its own particular electron or proton[7] which we may imagine colored to correspond to it. In our drawing of these spaces, we shall find any one ele-

[6] For minimization of notation we shall use dq to refer to the produce of the corresponding dq's for the x, y and z directions. In other words, $dq_s = (dq_s)_x(dq_s)_y(dq_s)_z$, and such a statement as q_s lying between q_s and dq_s is to be taken as symbolic of the statement that $(q_s)_x$, $(q_s)_y$, $(q_s)_z$ lie between the special values $(q_s)_x$, $(q_s)_y$, $(q_s)_z$, and $(q_s)_x+d(q_s)_x$; $(q_s)_y+d(q_s)_y$; $(q_s)_z+d(q_s)_z$. Moreover, we include protons in the general category of electrons.

[7] It seems doubtful whether one can include protons in the scheme in such manner that they contribute coordinates to the fundamental equation typified by (8). For, in that case, even for hydrogen there seems to be some difficulty in making the wave mechanics provide for a ψ function similar to that which is known to be satisfactory, and is obtained from a consideration in which the nucleus is regarded simply as a fixed origin. On the other hand, it is not

ment of volume duplicated n times in n different colors. If we assigned an electron to the red element of volume, and a blue electron to the superposed blue element of volume we could, as far as our ultimate thoughts concerning the disposition of electrons are concerned, dispense with the colors, and simply consider two electrons in that one element of volume of non-mysterious three dimensional space. The only reason for keeping our eye on the individuality of the electrons and their separate spaces is to provide a means for keeping the different colored coordinates distinct as regards the mathematical operations of integration and differentiation coming into the processes. Thus, for example, such a quantity as

$$dq_1dq_2dq_3\int\int\int\cdots\int\int\psi\bar{\psi}dq_4dq_5\cdots dq_n \tag{23}$$

has a perfectly definite meaning so long as we keep the different q's separate in our mind's eye. The integral is to be understood as taken simply with regard to the quantities q_4 to q_n not including q_1, q_2 and q_3. We dare not weld all the spaces together into one until we have performed the integration, as otherwise we shall lose sight of the significance of the integral. When the integration is completed, however, the result simply shows itself as

$$dq_1dq_2dq_3\phi(q_1, q_2, q_3, t)$$

and if we should assign

$$\begin{aligned}(q_1)_x &= 3, & (q_1)_y &= 4, & (q_1)_z &= 5\\ (q_2)_x &= 7, & (q_2)_y &= 9, & (q_2)_z &= 6\\ (q_3)_x &= 1, & (q_3)_y &= 6, & (q_3)_z &= 2\end{aligned}$$

the quantity would, as a matter of fact, represent the probability that electron No. 1 lay in the element of volume dq_1 of ordinary space, at the point $x=3$, $y=4$, $z=5$, and at the same time electron No. 2 lay in the element of volume dq_2 of ordinary space at the point $x=7$, $y=9$, $z=6$, and electron No. 3 lay in the element of volume dq_3 of ordinary space at the point $x=1$, $y=6$, $z=2$.

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easy to show that the desired end cannot be accomplished by a suitably chosen fundamental equation for ψ. We shall discuss the matter as though it can be accomplished. The only change that will be necessary if it cannot, will be to consider the coordinates of protons as eliminated from our equations, and leave these protons to be represented simply by a potential energy field, varying possibly with the time, but involving in its expression no coordinates other than those, associated with the electrons, and measured in relation to a fixed origin.

Now, in general it is of course our desire to find the probability of the presence of an electron, it matters not which electron, in some assigned element of volume. The probability that the electron No. 1 shall be in the element of volume dq_1 regardless of the positions of all the other electrons is:

$$\Delta P_1 = dq_1 \int\int \cdots \int\int \psi\bar{\psi} dq' \qquad (24)$$

where dq' refers to the product of all the dq's, including those corresponding to protons, but exclusive of dq_1. In view of the integrations, the quantity ΔP_1 is simply a function of q_1 multiplied by dq_1; and in view of the foregoing considerations ΔP_1 really represents the probability that the electron No. 1 shall lie in the element of volume $dxdydz = (dq_1)_x\ (dq_1)_y\ (dq_1)_z$ at the point $x = (q_1)_x$, $y = (q_1)_y$, $z = (q_1)_z$ of ordinary three-dimensional space; and in fact, we may write

$$\Delta P_1 = f_1(x, y, z)dxdydz$$

The probability ΔP_2 that electron No. 2 shall be in the element dq_2 regardless of the positions of the other electrons is

$$\Delta P_2 = dq_2 \int\int\int \cdots \int\int \psi\bar{\psi} dq'' \qquad (25)$$

where dq'' has a significance analogous to that of dq'. Hence, the probability that electron No. 2 shall lie in the element $dxdydz$ at $(q_2)_x = x$, $(q_2)_y = y$, $(q_2)_z = z$, is

$$\Delta P_2 = f_2(x \cdot y \cdot z)dxdydz$$

where $f_2(x \cdot y \cdot z)$ is the function obtained by replacing $(q_2)_x$, $(q_2)_y$, $(q_2)_z$, in the integral of (25) by x, y, z, respectively.

A similar result holds for all of the electrons. Thus, the probability ΔP that some unspecified electron lies in the element of volume $dxdydz$ at x, y, z, regardless of the positions of the others is[8]

$$\Delta P = \sum_\mu dq_\mu \int\int\int \cdots \int\int \psi\bar{\psi} dq_\mu^* \qquad (26)$$

[8] It might be thought that in addition to considering the probability of the presence of one electron in the volume element, we should also take account of the probability of the simultaneous presence of two, three, or more electrons in the volume element. In the limit, however, when the volume element is infinitesimal, the probability of the presence of more than one electron in the volume element is always infinitesimal compared with the probability of the presence of one electron therein.

where the sum is taken for all the coordinate elements dq_μ arising from electrons, and where, for each member of the sum, dq_μ^* refers to the product of all the dq's other than dq_μ, and ultimately, all the dq's outside of the integral are replaced by $dxdydz$, so that

$$\Delta P = dxdydz \sum_\mu f_\mu(x, y, z). \tag{27}$$

Once the ψ function is written down as a function of t and the coordinates, it is possible to proceed to the stage (27). In getting the ψ function, the starting point is the assumption of an appropriate differential equation, such as has been typified by (8). We are then to solve it, and pin the solution down to the actual problem we are interested in discussing. We do this by adjusting the arbitrary constants in the solution so as to correspond to the existence of the appropriate atoms, hydrogen, copper, or what not, and the various free electrons and protons, etc., which form our picture of the system at say, $t=0$. These systems are specified to us only through their ψ functions, and we must establish such correspondences between ψ functions and our pictures as will enable us to specify the latter in terms of the former. A helium atom is no longer to be specified as a certain planetary system. If we say that a single helium atom is to be at the point x, y, z at $t=0$, it means that the ψ function must be such as corresponds to one of the "eigen functions" characteristic of the helium atom, and that its center of coordinates must be at the point x, y, z at $t=0$. It is for us to say whether the atom is to be regarded as in its normal state or in one of its excited states; and the particular state it is in will then be symbolized by the appropriate "eigen function." The actual positions of the electrons in the atom are not to be specified further than to the extent that the assignment of ψ determines the probability of the presence of an electron in any assigned element of volume. For each atom or group of atoms in any assigned state there is a function which is to be regarded as characteristic of the system specified and of its state at the instant. The procedure for assignment of the appropriate function at $t=0$ in the case of a number of atoms mutually influencing each other is not so clear as regards its unique formulation. Every attempt to discuss, theoretically, the developments of an experimental situation necessitates the assignment of a correspondence between the actual observable things which constitute our apparatus, and some mathematical structure, or mechanical models which in our theory are to be the theoretical representatives of these material situations. On any classical atomic theory, the

mathematical analogy of a number of mutually influencing atoms could be sought by tracing the consequences of bringing the individual atoms together from infinity; and, an analogous method is open in wave mechanics, at any rate, in principle. We may, in fact, adjust the ψ distribution at $t=0$ to correspond to one which has developed out of another distribution prevalent at some earlier time, the characteristic of that earlier distribution being that it corresponded to a number of atoms very far apart. Its correspondence to this earlier situation would imply that it resulted in a sensibly zero probability for all configurations of the system in which the atoms were assigned in ordinary three-dimensional space to positions not very distant from each other.[9]

Suppose, as part of the initial conditions we have to express, we wish to take account of a beam of free electrons of certain assigned velocity emerging from a hole. Our original fundamental equation will now be amplified to the extent of as many extra trios of coordinates as there are electrons to be considered in the beam. Suppose that at $t=0$ we wish to make matters correspond to a situation in which the beam has just emerged as a cylindrical protuberance from the hole. Then our ψ function will have to satisfy the condition that, at $t=0$, expression (27) gives a finite probability for the presence of an electron in the cylindrical protuberance, and zero probability for all other regions, except insofar as it corresponds to the appropriate probabilities of the presence of electrons in the vicinities of the mutually influencing atoms in accordance with the ideas outlined above. To put the matter more precisely, our solution must be such that it may be regarded as having developed out of an earlier one in which the probability of the presence of an electron in one element of space was characteristic of the atomic system alone, while in another portion of space infinitely distant from it, the probability of the presence of an electron in an assigned element of the space was characteristic of the electron beam alone. Regarding the solution of the latter problem, however, there are certain elements which require further consideration, and arising from the fact that the system

[9] In actual practice, in such problems for example, as the reflection of electrons from a metal surface, it may be sufficient for many purposes to weld together the mutually influencing atoms which constitute the surface into one homogeneous entity whose properties for the purpose in hand become taken account of by the introduction of a suitably chosen potential function into the Schrödinger equation for the free electrons. Such simplifications are analogous to those in classical optics, where the highly complex story of the light entering a piece of glass from air is told as regards its main features by ignoring the atomic structure of the glass, and treating the glass merely as a homogeneous medium in which the light velocity is different from that in air.

is not one characterized by a definite and discrete set of constants E_m characteristic of the solutions of equation (8). Such a system is represented in its simplest form by a group of free electrons, or even of electrons in certain types of electromagnetic fields. Thus, for example, the customary form of the equation for ψ corresponding to n free electrons is

$$-\frac{h}{2\pi i}\frac{\partial\psi}{\partial t}=\left(\frac{h}{2\pi i}\right)^2\left(\frac{1}{2m}\right)\sum_s\left\{\frac{\partial^2}{(\partial q_s)_x{}^2}+\frac{\partial^2}{(\partial q_s)_y{}^2}+\frac{\partial^2}{(\partial q_s)_z{}^2}\right\}\psi\cdots \quad (28)$$

which is the form based upon the Hamiltonian function

$$H=\frac{1}{2m}\sum_s\left\{(p_s)_x{}^2+(p_s)_y{}^2+(p_s)_z{}^2\right\}. \quad (29)$$

It is not difficult to show that the solution for this case is

$$\psi=\sum_\delta\int\int\int\int A_{\alpha\beta\gamma}\,e^{-2\pi i/h\left[Et-(2m)^{\frac{1}{2}}\left\{\alpha q_{sx}+\beta q_{sy}+\gamma q_{sz}\right\}+\delta_{\alpha\beta\gamma}\right]}\cdots d\alpha d\beta d\gamma \quad (30)$$

where $E=\alpha^2+\beta^2+\gamma^2$ and where $A_{\alpha\beta\gamma}$ and $\delta_{\alpha\beta\gamma}$ are arbitrary functions of α, β, γ, to be determined by the boundary conditions assigned. The integration with respect to α, β, γ, represents really the sum of an infinite number of solutions analogous to the ψ_m referred to above, with the inclusion of the arbitrary constants $A_{\alpha\beta\gamma}d\alpha d\beta d\gamma$.

Suppose, for example, we have a beam of free electrons of certain assigned velocities emerging from a hole; and, suppose that at $t=0$, we wish to make matters correspond to a situation in which the beam has just emerged as a cylindrical protuberance from the hole. Then our ψ function will have to satisfy the condition that at $t=0$ expression (30) gives a finite probability for the presence of an electron in the cylindrical protuberance, and zero probability for every other region. Or speaking more generally, the expression for ΔP given by (27) at $t=0$ must correspond to the assigned probability distribution at that instant. Now on working matters out, it results that the assignment of ΔP at $t=0$ is sufficient to determine $A_{\alpha\beta\gamma}$ and $\delta_{\alpha\beta\gamma}$ as functions of α, β, γ, and ψ becomes therefore completely determined.[10] This is at first sight rather surprising, because, in completely determining ψ the whole problem has been implicitly solved including all that is to be said about the distribution of the velocity of the electrons; and, it seems strange that the assignment of ΔP, which is the equivalent of the assignment of the

[10] The complete details pertaining to this matter will be given elsewhere as they would lengthen the present article in undue proportion in comparison with the other matters here discussed.

average charge density at a point should result in a determination of the velocity distribution which we might suppose, on experimental grounds, to be independently assignable. On closer examination, however, the matter does not seem so strange, but in fact, quite natural. A wide enough generality to illustrate the salient features of this matter will be obtained by restricting ourselves to the case of one dimension, so that the summation in (30) disappears, and we then have

$$\psi\bar{\psi} = \int\int\int\int\int\int A_{\alpha_m\beta_m\gamma_m} A_{\alpha_n\beta_n\gamma_n} e^{-2\pi i/h\left[(E_m-E_n)t-\left\{(2m)^{1/2}(\alpha_m-\alpha_n)x+(\beta_m-\beta_n)y+(\gamma_m-\gamma_n)z\right\}+\delta_{\alpha_m\beta_m\gamma_m}-\delta_{\alpha_n\beta_n\gamma_n}\right]} d\alpha_m d\beta_m d\gamma_m d\alpha_n d\beta_n d\gamma_n . \quad (31)$$

We must also here remark that the quantity which becomes associated with the average velocity of an electron at a point is w, where,

$$\psi\bar{\psi}w = \frac{h}{4\pi i m}\{\bar{\psi}\,\text{grad}\,\psi - \psi\,\text{grad}\,\bar{\psi}\}\,.^{11} \quad (32)$$

Applying this to (30) we find, for w_x for example

$$w_x = \frac{1}{(2m)^{1/2}}\int\int\int\int\int\int A_{\alpha_m\beta_m\gamma_m} A_{\alpha_n\beta_n\gamma_n}(\alpha_m+\alpha_n) e^{-2\pi i/h\left[(E_m-E_n)t-(2m)^{1/2}\left\{(\alpha_m-\alpha_n)x+(\beta_m-\beta_n)y+(\gamma_m-\gamma_n)z\right\}+\delta_{\alpha_m\beta_m\gamma_m}-\delta_{\alpha_n\beta_n\gamma_n}\right]} d\alpha_m d\beta_m d\gamma_m d\alpha_n d\beta_n d\gamma_n .$$

If then we wish to assign w_x, at $t=0$, this assignment places a control upon the A's, and so upon the assignment of ΔP as a function of x, y, and z at $t=0$. However, different assignments of the velocity does not in all cases result in vastly different distributions for ΔP. Reverting for a moment to classical optics, we recall how, by the superposition of waves with suitably adjusted phases we can account for a parallel beam of light whatever the frequency of the light may be. Outside the parallel beam there will be an interference pattern which does depend upon the frequency of the waves. Indeed, if we express precisely the distribution of energy at all points in space, then not only the phase relations but also the frequencies of the waves will become determined. In fact, if there is present a multitude of frequencies, a *complete* specification of the distribution of energy at all points in space will serve to determine all

[11] It can be shown that minus the divergence of the right hand side of (32) multiplied by the electronic charge is equal to $e\partial/\partial t(\psi\bar{\psi})$. In other words, this value of w preserves the equation of continuity for charge represented by a density equal to $e\psi\bar{\psi}$.

the phase relations and the wave amplitudes associated with each of the frequencies. A specification of energy density distribution which calls for no more than the energy in the beam, and which neglects what we may crudely call diffraction phenomena may be provided for by any frequency or any mixture of frequencies. And so, in the wave mechanics problem, it turns out that while an exact specification of ΔP as a function of x, y, z, at $t=0$ carries with it a complete determination of the A's and δ's and so a complete determination of all that may be said as regards the average velocity, a specification which calls simply for a parallel beam of electrons of a certain specified distribution of ΔP at $t=0$, and does not concern itself with the interference pattern is one which permits *any* average velocity to be associated with it. We could, for example, confine ourselves to a single value of E, and with the latitude available in α, β, γ, permitted by

$$E = \alpha^2 + \beta^2 + \gamma^2$$

we could by (30) as applied to this case provide for an approximately parallel beam of ψ waves of limited cross section emerging from a hole.

With the problem of the boundary conditions settled, the whole story of all that may be asked is told in the solution of the appropriate ψ equation subject to prescribed boundary conditions. For that solution provides the means of calculating the probability that an electron or proton shall be present in any assigned element of volume of ordinary three dimensional space; and, in the last analysis this is all that may be asked.[12] In the last analysis there is no light. There are no x-rays, no quanta, except insofar as it may be convenient to introduce these concepts as part of a subsidiary scaffolding to facilitate the mathematical technique of the general theory. A purely idealistic view of the situation would, indeed, take us into rather a strange viewpoint as regards all experiments. If an experiment concerns itself with the entry of electrons into a Faraday cylinder, we should really not stop there, but allow the wave mechanics to tell its own story of the action of the Faraday cylinder, of the electrometer attached, of the deflection of the spot of light and so forth. Indeed, a logical pursuit of this line of argument would lead us eventually to the brain itself, and would involve a wave mechanical theory of the mental phenomena leading to the conclusions which

[12] A reservation should perhaps be made in the matter of such experiments as those of A. H. Compton and A. W. Simon (Phys. Rev., V. 26, pp. 289–299, 1925) in which it is possible that a complete theory may be expected to require that when a primary x-ray is scattered by a single atom, the scattered x-ray and recoil electron are related in a perfectly definite manner.

we write down in our notebooks, the final step being the setting up of a correspondence between some ultimate expression of the results of the experiment (such as the words written down in our notebooks) and the probability of certain electronic configurations associated, possibly, with the brain. However, we need not be as pedantic as all this; and may, for practical purposes, short circuit most of these complexities by terminating the wave mechanical argument at some point where the probability of a certain electronic configuration has been calculated, and, then complete the argument by classical, physical or chemical means in those parts in respect of which we have no doubt as to the approximate applicability of such means.

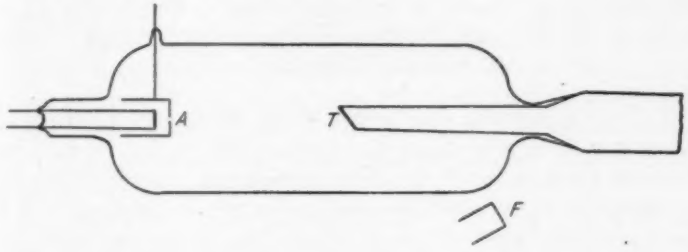

FIG. 2

It is worthy of notice that, in the spirit of the foregoing attitude towards wave mechanics, there remain none of those difficulties concerned with the propagation of waves in a multidimensional space, difficulties which really do not belong to the subject at all, and which find their origin in an attempt to retain the idea of a wave propagation in a medium.

In order to illustrate the significance of the foregoing remarks, let us suppose that we are presented with a problem, which in the language of the experimental physicist, would be described as follows: A beam of 100,000 volt electrons comes through the hole A, Fig. 2 and strikes the target T, producing x-rays. Some of the x-rays fall upon the plate J from which they eject electrons, some of which are caught in the Faraday cylinder F. We may further complicate the experiment in supposing that the heating of the plate J by the x-rays is so great that some of that atoms evaporate from the plate and condense on the x-ray

tube. It is desired to inquire of wave mechanics as to the story of the influx of electrons to the Faraday cylinder, and the evaporation of the substance of the plate J on to the glass of the x-ray tube.

Our first step must be to write down a differential equation for ψ. In the stage to which wave mechanics seems so far to have crystallized itself in the matter of writing down this equation the plan is to seek the Hamiltonian function H which would be appropriate to the discussion of the electrons, x-ray target, plate J, etc., as one system on the basis of the old mechanics and then after replacing any momentum p_r, by $h/2\pi i \cdot \partial/\partial q_r$ treat the resulting expression for H as an operator which operates on ψ, and write for our differential equation

$$-\frac{h}{2\pi i}\frac{\partial \psi}{\partial t} = H\left(\frac{h}{2\pi i}\frac{\partial}{\partial q_1}, \frac{h}{2\pi i}\frac{\partial}{\partial q_2}, \cdots q_1, q_2, \cdots t\right)\psi.$$

We shall, for a moment, defer any criticism of the significance of this procedure, and rest our attention on the supposition that we have *some* differential equation for ψ. It is now our business to solve it subject to the appropriate initial conditions which must be taken account of in the manner already described. In particular, the initial conditions must take account of the beam of 100,000 volt electrons specified as part of the experiment. The initial conditions of the problem having been expressed, the remainder of the story is to be told by the solution of the differential equation. It will automatically result that ψ as a function of the q's for the target will remain sensibly what it has always been until such time as is represented by what on classical theory would be the time of arrival of the electrons there, when it should begin to change rapidly.[13] The experimentalist will want to talk about the emission of x-rays. However, we must not listen to him, but must let ψ tell its own story by gradually propagating its alterations until the regions at J begin to experience such alterations. The time rates of change of $\psi\bar{\psi}$, and through it of the quantity ΔP of equation 27 should then contain in itself the story of the passage of the electrons from the plate and of their appearance in the Faraday cylinder. It must tell this by ΔP becoming

[13] More exactly we should say that, initially the value of ψ would be sensibly the same as that pertaining to a case where the electron beam was absent, for all values of each of the coordinates other than those values which if represented in three-dimensional space would correspond to a position near the origin of the beam. Moreover, the value of ψ would remain sensibly the same function of the q's and of t for such values of the q's as correspond to points near the target until the lapse of a time approximately equal to what on classical theory would be the time taken by an electron in travelling from the origin of the beam to the target.

finite for elements of volume within the Faraday cylinder whereas at $t=0$, and for sometime subsequently it was zero there. The story of the evaporation of the metal from the plate J and its deposition on the glass tube would be told by the ψ function automatically changing in such manner as to assume a form appropriate to the existence of atoms of the metal on the tube with fewer atoms on the plate T. If the experimentalist should place a stopping potential V on the Faraday cylinder and should seek to measure the velocities of the electrons by adjusting V so as to just keep them out, the wave mechanics story must be that of including V in the original differential equation, and subsequently determining for it a limiting value which just insures that the quantity ΔP referred to above is zero for all time for points inside the Faraday cylinder. If, in the experiment, the glass tube of the x-ray were replaced by lead, the lead would also figure in the Hamiltonian function, and in such a way as to cause the solution for ψ outside of the tube to be sensibly independent of all boundary conditions expressed for the regions inside the tube. In this manner the story of the shielding action of the lead would be told.

Now, I fear that I have not been able to suppress light from this picture without leaving you to some extent in the dark. You will naturally have a suspicion that all of those characteristic properties of optics and x-rays on which so much time has been spent cannot evaporate without giving any account of themselves. Of course, I can adopt the lordly pedantic attitude of telling you that they should all be contained, somehow or other in the fundamental equation for ψ. There is, however, a sufficient richness of content in these optical or x-ray phenomena to warrant your asking me *how* they are contained in the differential equation for ψ.

Now the mathematical physicist who did not trouble himself very much about the logic of the situation, was accustomed to proceed in a way different from the ideal one which I have cited, a way which, in the case of the present problem, may be symbolized as follows: He took a beam of plane ψ waves, coming from the region of the hole, and representing according to the conventional principles of the theory, a single electron, the velocity of the electron being represented by a suitably chosen value for the wave length of these ψ waves. Then, a solution for ψ was sought which reverted to the same value as for the wave at large distances from an atom of the plate T, and which when the parameter which expressed the amplitude of the plane wave was made zero reverted to the form appropriate to the said atom, or rather to the form

appropriate to the potential distribution which in the classical theory would be associated with the atom in question. The resulting calculation of $\psi\bar{\psi}$ then provided a fictitious density distribution which served to define in the space around, a set of electromagnetic vectors, representing the x-radiation, for example. The electromagnetic field resulting from this x-radiation was then incorporated as a correction on the Hamiltonian function of some other atom where it was of interest to discuss the effect of the x-rays. It was incorporated in the old classical manner, and a new Schrödinger equation was set up at the seat of operation of the x-rays for the purpose of discussing what happened there, for discussing, for example, the photoelectric emission of an electron, or the Compton effect, and so forth. Now while this procedure does not, on the face of it, seem to have any very firm logical background unless it can be shown that it is the analytical equivalent of the more ideal procedure which I have already sketched, the fact remains that it was successful in fitting the facts, the facts of photoelectric emission and so forth. Of course, the story which might be expected of the ideal procedure is uncertain until more has been said as to the precise nature of the original equation for ψ. It remains then to attempt to formulate some respectable ancestry for the procedure which is found to fit the facts so as to render it a certain analytical equivalent of the more ideal theory. In the ordinary treatment of a dynamical system subjected to a perturbation by an external electromagnetic field, the method of incorporating that external field in the Hamiltonian function for the system, is well understood. Suppose then, we take this Hamiltonian function, modified in this way, and use it in equation (8), but suppose that the electric and magnetic fields, or rather the so-called scalar and vector potentials which make their appearance in the Hamiltonian function in this way, are understood as calculable not from the positions and motion of the real electrons even though we should know them, but from the $\psi\bar{\psi}$ function itself, or rather from the quantity $f_\mu(x, y, z)$ of equation (27) treated in the role of what in the classical theory would be a charge density.[14] Our equation (8) will then assume a form which I may designate as

[14] The conservation of logical precision in the various parts of the theory really necessitates that we divide ΔP into a number of parts corresponding to the various terms of the sum in (27). Each of these contributions may be thought of loosely as associated with one of the electrons. With each of these parts we then associate a velocity somewhat similar to (32) for the one dimension case, but actually of the form given by

$$w_\mu f_\mu(x, y, z) = \frac{h}{4\pi i m} \int \cdots \int (\bar{\psi}\ \mathrm{grad}_\mu\ \psi - \psi\ \mathrm{grad}_\mu\ \bar{\psi}) dq^*$$

$$-\epsilon\frac{\partial\psi}{\partial t} = (H_1 + H_2)\psi$$

in which H_1 is the portion of the Hamiltonian operator which we should have in the absence of the extra terms, and H_2 is the part which involves these extra terms, and which will represent contributions from all points in three-dimensional space in the form of space integrals. We can, of course, if we wish, write a corresponding differential equation for the quantity $\bar{\psi}$, and these two equations as they now stand, will represent formally, differential equations for these quantities, ψ and $\bar{\psi}$. We might adopt the formal procedure of solving them, and the solution of our problem would then take what I have called the ideal form. On the other hand, in such a problem as that which I have cited and represented in Fig. 2, we can see, or at any rate have a strong suspicion that certain approximations may be made which bring this exact procedure down to an equivalent of the procedure actually adopted in these problems. For, calling upon our intuition for a moment, we feel convinced subject of course, to subsequent confirmation, that the true value of ΔP for regions in the vicinity of the plate T is practically the same as would have been obtained from a ψ function which was built upon the basis of a differential equation which took account of nothing but the electron beam and the atoms of the plate T. We may go further and say that we feel convinced that even in this calculation we may neglect terms depending upon the operator H_2. Again, considering values of ΔP in the region of the plate J, we feel convinced, subject to subsequent analytical confirmation, that the values of ΔP in question are practically the same as would be obtained from a ψ function built up on the basis of a differential equation which took account of nothing but the plate J itself, and the electromagnetic terms (they are scalar and vector potentials), in the Hamiltonian operator for that case. These electromagnetic potentials are calculable as contributions from each volume element of three dimensional space; and, subject again to subsequent analytical confirmation, we feel convinced that the only contributions which are significant will be those calculable from the values of $f_\mu(x, y, z)$ and the associated velocities applicable to the problem in which the approximate solution corresponding to the electron beam and the plate T above were considered. Such a procedure represents the equivalent of that which I have referred to above as deficient in obvious logical background. My

where w_μ is the velocity associated with the contribution to ΔP by the term $f_\mu(x, y, z)$ in (27) with subscript μ, where dq^* has the significance implied in equation (26), and where grad_μ refers to the gradient calculable in the space with coordinates $(q_\mu)_x$, $(q_\mu)_y$, $(q_\mu)_z$.

purpose in this rather involved and halting argument is to show how we may think of the problem in such a manner as to make use of our previous experience and intuitions for the purpose of suggesting approximate methods, and yet have the problem sufficiently intact in its exact sense so that we may investigate the validity of the approximations if we so wish by substituting, subsequently, in the exact equations the approximate expressions so obtained.[15] A significant element of the procedure is that the quantity $f_\mu(x, y, z)$ which figures as part of an effective charge density in the calculation of the vector potentials is a function of the three coordinates in ordinary three dimensional space regardless of any mathematical artificiality in the coordinates associate with the particular ψ function from which $f_\mu(x, y, z)$ was calculated.

Of course, in a procedure of this kind, the actual physical reality of light and x-rays seems to some extent subordinated to the position of a sort of scaffolding in the whole mathematical structure. There is law and order in that scaffolding, and indeed such a richness of law as to make a part of the scaffolding stand out by itself with a prominence sufficient to demand its own picture. The cynically minded will have some ground for asserting, however, that there is no light, and there are no x-rays; or, at least that the quantity which has the greatest claim to the term reality, the greatest claim to having an aether in which to ingrain its existence is the quantity $\Sigma_\mu f_\mu(x, y, z)$. Light and x-rays are simply the expression of the way in which this quantity governs the relationships of its value at different points in space time, and the electromagnetic equations are simply the analytical tools convenient for the expression of that regularity.[16]

One may well wonder what has become of the electron itself, in all of this discussion. It seems to have lost the only job it had, the job of

[15] The approximate expression for ψ would be represented as the sum of a number of parts. Thus, for example, there would be ψ_T calculated in connection with plate T, and expressible as a function of one set of coordinates. There would also be ψ_J calculated in connection with the plate J, and expressible as a function of another set of coordinates. The complete differential equation for ψ would involve all the coordinates and it would be necessary to show that the approximate value of ψ given by $\psi=\psi_T+\psi_J+\cdots$ when substituted in the equation $+\epsilon\,\partial\psi/\partial t+(H_1+H_2)=0$ gave for the left hand side a result which if not zero, was small compared with the magnitudes of the larger terms of which it was composed.

[16] Within the limits of this address it has been impossible to say anything concerning the position occupied by electromagnetic phenomena in the theory of Dirac. My object has been simply to speak of certain matters in the older wave mechanics aspect of electrodynamics in the hope of rendering clearer a few of the fundamental ideas connected with types of calculation which have played a large part in the solution of problems so far attempted in wave mechanics.

producing an electromagnetic field, which duty has been gobbled up by the omniverous ψ, and its offspring $\Sigma_\mu f_\mu(x, y, z)$. When a finite value of $\Sigma_\mu f_\mu(x, y, z)$ has been calculated for points in the Faraday cylinder we are content to say that a certain deflection will occur in the electrometer connected to it. We can get on without the electron if we like, and speak only of $\Sigma_\mu f_\mu(x, y, z)$. The electrons are simply symbols. They are counters which we throw into the Faraday cylinder to register by their numbers the numerical value of this quantity there.

And now, in having reduced the whole of light, x-rays, and the like to such a mystical position, I feel that I owe an apology to the Optical Society in particular if I appear to have contended that the fundamental agency of its science is but a dream. However, I am not flattering myself that the Society will go out of existence after this lecture, or that it will become impossible to buy any more spectacles. For I am conscious of the fact that anyone in the audience could anticipate the conclusion which I now make of this address by going outside and turning off a switch associated with certain wires which I believe go to these glass bulbs that I see overhead.

Matter and Radiation, with particular reference to the detection and uses of the infrared rays. By John Buckingham, Assistant Director of Scientific Research for the Admiralty (London), xii+144 pages. Oxford University Press; London: Humphrey Milford. $4.00.

This book is an elementary yet very readable discussion of the general problem of radiation and its relation to matter. "In order that the non-scientific reader may be relieved from the initial over load of strange words" the author has avoided, "so far as possible, the use of terms which are not a part of the vocabulary of every day life." The author warns the reader, however, that even so the ideas presented by the book will not be acquired casually but will be understood only after careful reading. In the opinion of the reviewer the book is quite worthy of such careful reading.

The book is divided into five chapters. The first, entitled "Electric Waves," begins with a discussion of our early ideas concerning light; developes the idea of wave motion in part by means of acoustic analogies; shows the necessity of the concept of an "ether" as a carrier of electric and magnetic forces; and then discusses Maxwell's theory.

Chapter II, discusses atoms and molecules with particular reference to our present concepts of the nature of these structures. The reasons for our belief in the electrical nature of matter are pointed out.

Chapter III, "The Electrical Structure of Atoms and the Production of Radiation" begins with a discussion of the Rutherford-Bohr atom model. The production of light by hydrogen is then discussed and the general idea of the quantum theory is introduced. There is a description of the different ways in which the various parts of the spectrum from x-rays to infrared are produced.

Chapters IV and V deal with infrared rays, their detection, and uses. The various ways of observing and studying infrared are discussed—such as photography, thermoelectric phenomena, radio micrometers, etc. A large section of the book—some thirty pages—is devoted to applications of infrared rays for such purposes as the detection of invisible objects, secret signaling, burglar alarms, fog penetration, including photography through fog, and the general question of the absorption of light and infrared rays by fog and mist. The author expresses regret that so many of the applications of infrared rays which he discusses are of military use. His discussion, however, makes it clear that these are illustrations only and should be equally applicable to every day life.

It is unfortunate that the author has not seen fit to prepare an index. Even though the book is intended for the lay reader, it will probably serve as a reference book and an index would have been exceedingly useful.

F. K. RICHTMYER

Structure of Line Spectra. L. Pauling and S. Goudsmit; x+263 pp. McGraw-Hill Book Co. Inc., New York. $3.50.

This, the third volume of the International Series in Physics, will fill a long-felt need. While certain portions of the book are rich in experimental fact, the general policy is to discuss broad regularities and to illustrate them briefly, rather than to present large amounts of empirical detail. In regard to difficulty, the book is well suited for a half-year's graduate course, and probably could be read with profit by some seniors.

Chapter I contains a brief statement in regard to the series of hydrogen and describes the vector atomic model of the older quantum theory. The authors point out the desirability of modifying the magnitudes attributed to the vectors of this model, in such a way that it will yield most of the results obtained on treating the classical atomic model by wave methods. This chapter may be considered as a statement of the purpose and general trend of the book and therefore will be useful to those who know the subject, but considered as an introduction for beginners, it must be said that much of the material would not be understood. Chapter II, dealing with stationary states of the hydrogen atom, presents the Bohr theory of elliptic orbits. Some of the equations on Pages 22 and 23 are unfortunately in error but the necessary modifications will be obvious to most readers. The Schroedinger wave equation is introduced and solutions for hydrogen are obtained. The treatment is too brief for beginners but for advanced readers it will serve as an excellent review of the principal facts in regard to the characteristic energy values and functions of hydrogen. Another paragraph is devoted to the modified vector model. Term values for penetrating and non-penetrating "orbits" of alkali-like atoms are next discussed. Elementary facts concerning absorption spectra, selection rules, term diagrams, and critical potentials, are briefly presented. Chapter IV deals with the explanation of single electron spectra by the aid of the electron spin. No attempt is made to explain the ideas of Dirac in regard to the origin of the spin, although the matter is briefly mentioned. This chapter is very pleasing, due to the authors' straightforward method of introducing the spin without first beclouding the issue by the discussion of the older relativity theory of doublet spectra. The Zeeman effect and Stark effect of single-electron spectra are introduced in Chapter V, while Chapter VI applies the modified vector model to atoms with two valence electrons. Here it is necessary to consider resonance energy, and the writers say: "In order to continue to make use of the vector model, the fiction may be adopted that, as a result of some effect not represented in the model, there exists a great interaction energy between the two electron spins; with this assumption a complete description can be given of the structure of spectra. It must not be forgotten, though, that this interaction energy is in reality the energy of the Heisenberg-Dirac reso-

nance phenomenon, and is to be attributed in reality to the charges of the two electrons and not to their magnetic moments." This chapter also deals empirically with selection rules, metastable states, the equation for triplet separations, and Heisenberg's treatment of triplet structure in light atoms. The various types of coupling are briefly surveyed. The chapter on the vector model for many electron atoms presents the essential facts quite lucidly, emphasis being placed, of course, on the most important case, namely, Russell-Saunders coupling.

Chapter VIII on the intensity and polarization of spectral lines would be better if considerably amplified. In discussing Einstein's A's and B's the elements of the coordinate matrices are introduced with very little explanation and experimental facts are almost absent. It is stated that "the actually observed intensity contains also the fourth power of the frequency of the spectral line as a factor. Only in comparing the intensities of lines which lie near together in the spectrum is it allowed to neglect this factor." Actually, the experimental situation in regard to this correction factor is a complicated one, and this fact might well be developed in some detail. This is only one of a number of cases where statements are made without sufficient explanation of their limitations. No one realizes more keenly than the reviewer how difficult it is to present briefly a subject like the theory of spectra, where most laws have many exceptions. The point he wishes to make is that statements possessing this characteristic should be amplified and illustrated regardless of the additional space required, or that they should be hedged with qualifying phrases.

Chapter IX introduces the Pauli exclusion principle and its important consequences for the periodic system. The treatment of the Stoner subgroups is valuable and the discussion of permanence and sum rules is detailed and useful. The material on the relative positions of spectral terms might well be extended and illustrated by reference to spectral data. The book seems to increase in interest to the research man as one goes on. Chapter X dealing with x-ray spectra, has an excellent account of the various screening constants and the equations which they obey. The material on multiplet spectra in the x-ray range is too brief to be very useful, but extensive references are given. Also, the treatment of x-ray spectra of higher rank is so brief that it is likely to leave the reader with an inadequate impression of the real complexity of these phenomena. The concluding paragraph on the application of x-ray laws to optical spectra is very good. Chapter XI on hyperfine structure and nuclear moments is very up-to-date and replete with experimental facts. The concluding chapter discusses the Stern-Gerlach experiment and paramagnetism, preparatory to a satisfactory treatment of magneton numbers. The book ends with a section on the polarization of resonance radiation in the magnetic field. The appendices contain useful numerical data, detailed descriptions of the

spectra of iron and neon, and a table showing the allowed states with (jj) coupling.

In conclusion, a second edition should be much longer and should contain more experimental detail and wave-mechanical derivations. All in all, this book is the most detailed and up-to-date account of line spectra which we possess, and is written as clearly and logically as the difficulties of the subject permit.

ARTHUR E. RUARK

New Frontiers of Physics. By Paul R. Heyl, Bureau of Standards Washington, D.C. 170 pages. D. Appleton & Company, New York. $2.00.

This little book initiates the "Appleton New World of Science Series," written for "the intelligent general reader without special training who wishes to read timely and vivid accounts of the exciting new developments in science." It sets forth some of the present day questions of physics, with enough of the historical setting to indicate how the questions came to be sensed.

"Matter was a sharply defined . . . . but a lifeless concept" of the 18th century; "energy was distinctly a 19th century concept"; and in the same century the ether came to be felt necessary to the transmission of energy.

Under "Matter and Its Structure" are discussed the 20th century atomic models and theories associated with the names of Thomson, Rutherford, Bohr, De Broglie, Schrödinger, in all of which developments appears "the idea that matter was not a fundamental entity, but a mode of manifestation of something else."

"Energy and Its Structure" brings in the quantum theory,—"a lusty infant, though growing rapidly."

The interchange of energy and matter is considered as applied to the sun's heat, to the source of the cosmic rays, and to radioactive phenomena.

It is in the chapters on "Space and Time" and "Gravitation and Cosmology" that the reader meets the puzzles and their puzzling answers offered by Einstein and relativity. The lay reader will be entertained, amused, or enthused, according to his attitude. He may possibly retain some of the terms for later use in conversational display of the sort that popularized science encourages,—a use that is doubtless good for science if the conversationalist remains conscious of his actual ignorance of the reasons for the conclusions he may mention so glibly. Dr. Heyl's discussion will at least make him feel on speaking terms with the Michelson-Morley experiment, relativity, four-dimensional space, curved space, and world lines, and may awake a desire for a more intimate acquaintance.

Calling the theory of relativity "a hollow mathematical shell with no real content; useful where it fits the facts, useless where it does not,"

the author offers comfort to the relativists when, discussing Professor Miller's positive ether drift, he says, "Michelson's recent result (1928) confirms the negative conclusion of the original experiment, and has been accepted by Professor Miller." The implication in this statement does not confirm the reviewer's information as to Professor Miller's position regarding his ether drift data.

Dr. Heyl closes with a reference to the present trend of physical science toward "consolidation and correlation of what were at one time regarded as independent phenomena," and the "steady drift away from materialism in fundamental concepts."

The book is full of striking figures of speech that please and stimulate, helping the reader to see by analogy what certain abstract concepts may be, or at least what they may be thought like.

HARLEY HOWE